IQI 04, Seminar 4

Produced with pdflatex and xfig

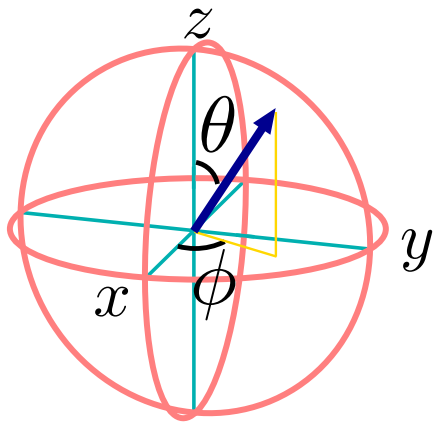
- One qubit rotations.
- Universality for one qubit.

E. “Manny” Knill: knill@boulder.nist.gov

Rotations I

- Rotation gates rotate the state in the Bloch sphere representation.

$$\alpha|0\rangle + \beta|1\rangle \cong e^{-i\phi/2}\cos(\theta/2)|0\rangle + e^{i\phi/2}\sin(\theta/2)|1\rangle$$



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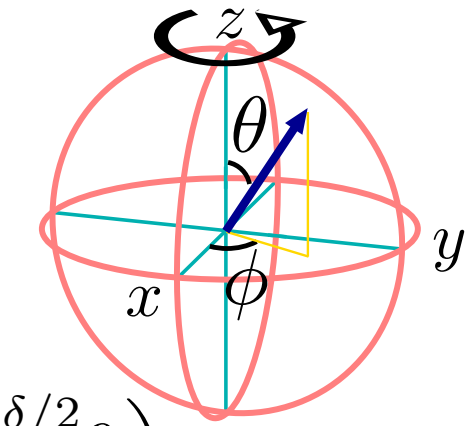

- Z-rotations.

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \xrightarrow{\mathbf{Z}_\delta} \begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix} e^{-i\delta/2}\alpha \\ e^{i\delta/2}\beta\end{pmatrix}$$

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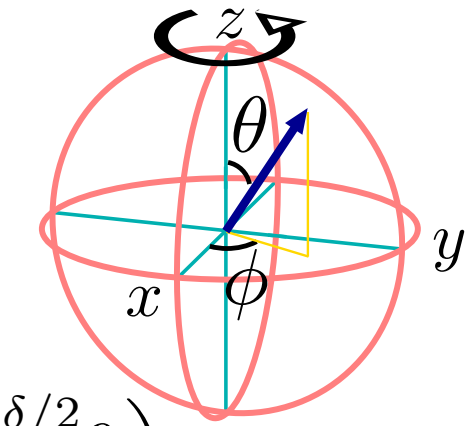

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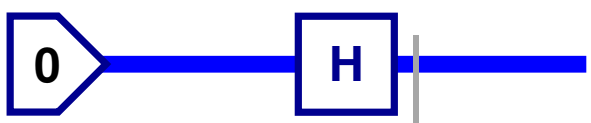

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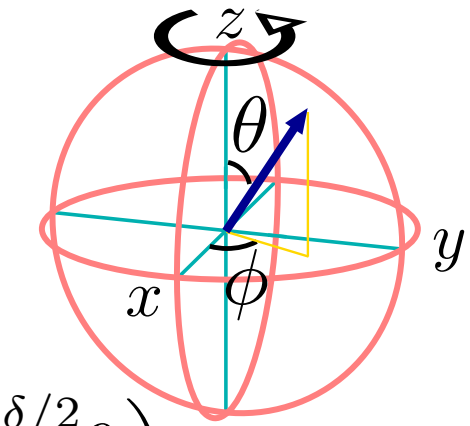

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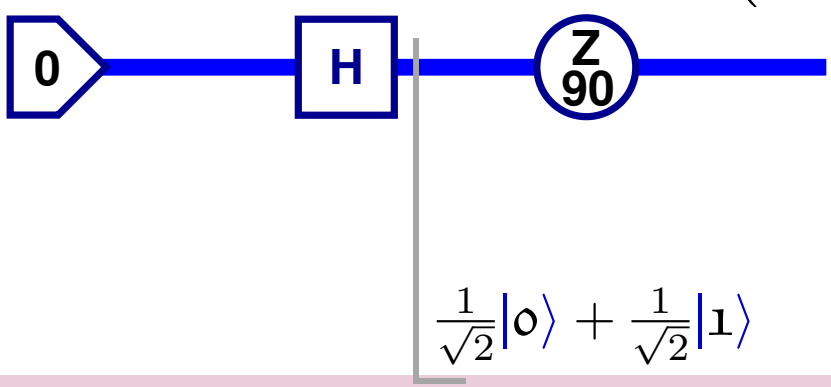

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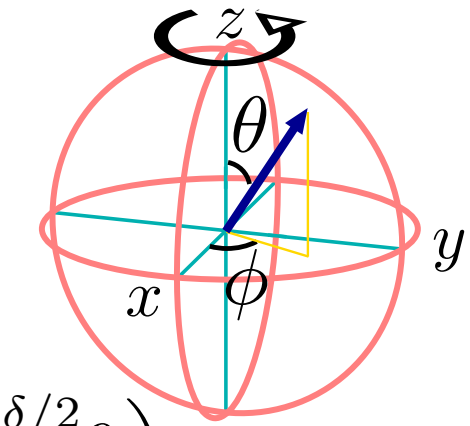

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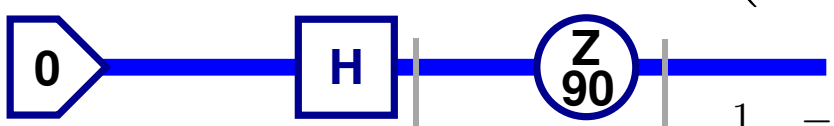

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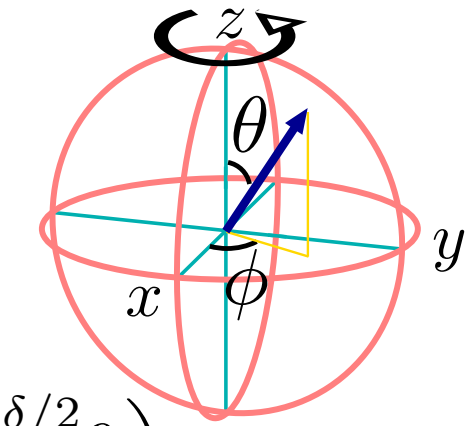

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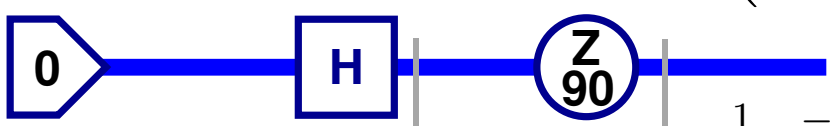

$$\frac{1}{\sqrt{2}}e^{-i\pi/4}|0\rangle + \frac{1}{\sqrt{2}}e^{i\pi/4}|1\rangle = \frac{1}{\sqrt{2}}e^{-i\pi/4}(|0\rangle + i|1\rangle)$$

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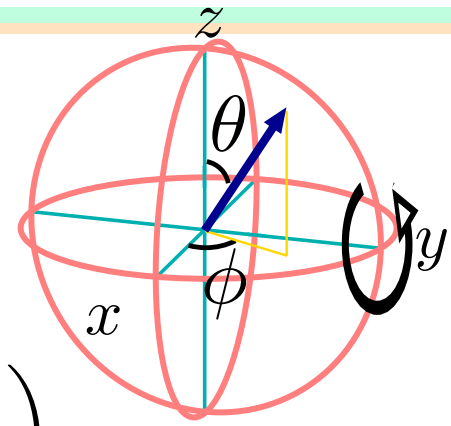

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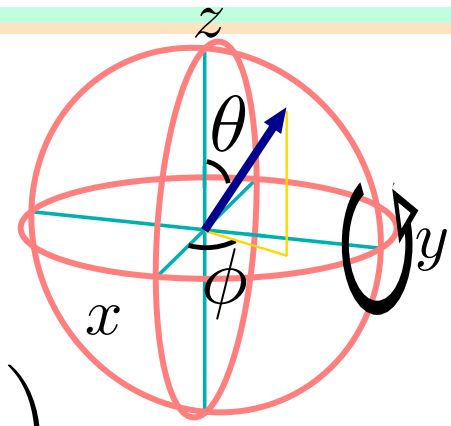

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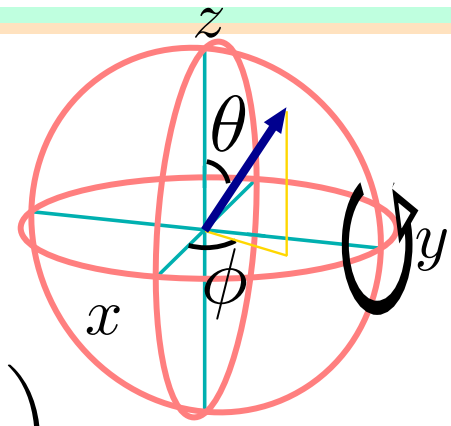

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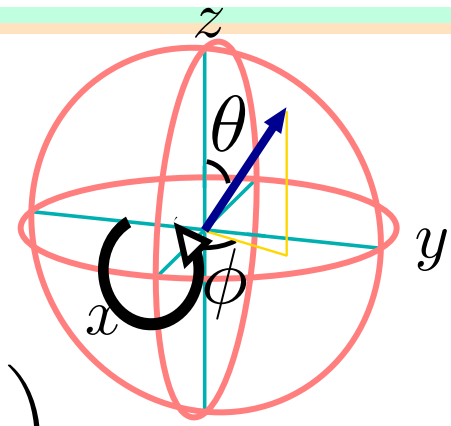

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A One-Qubit Network

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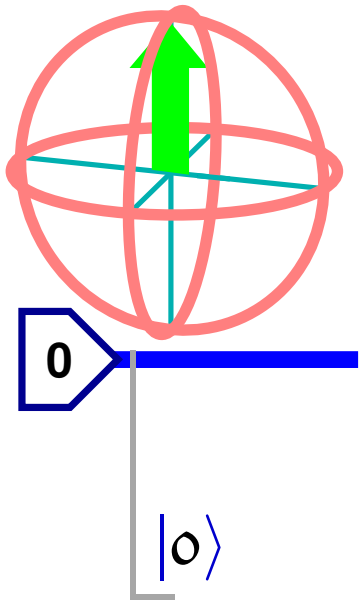

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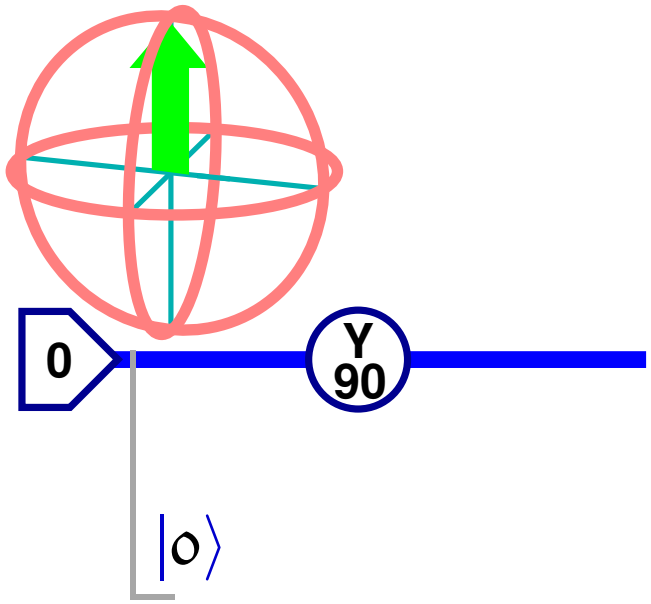

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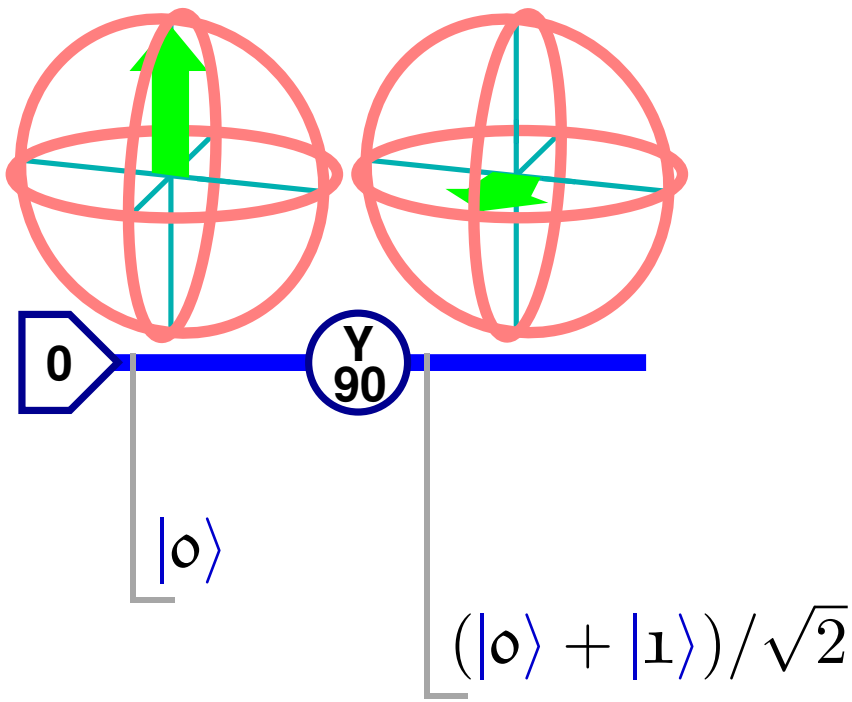

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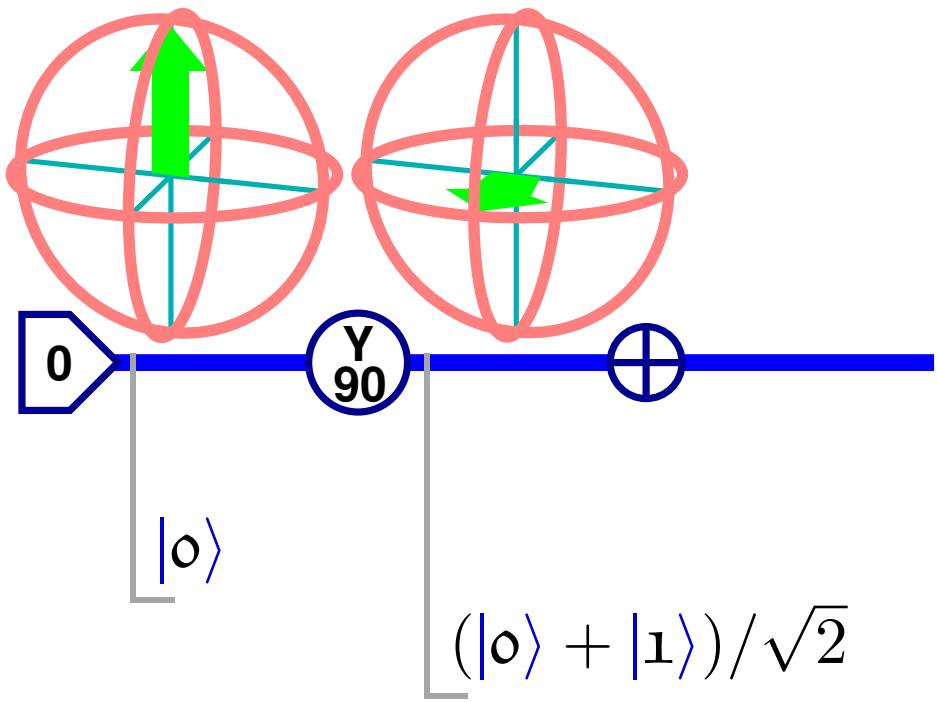

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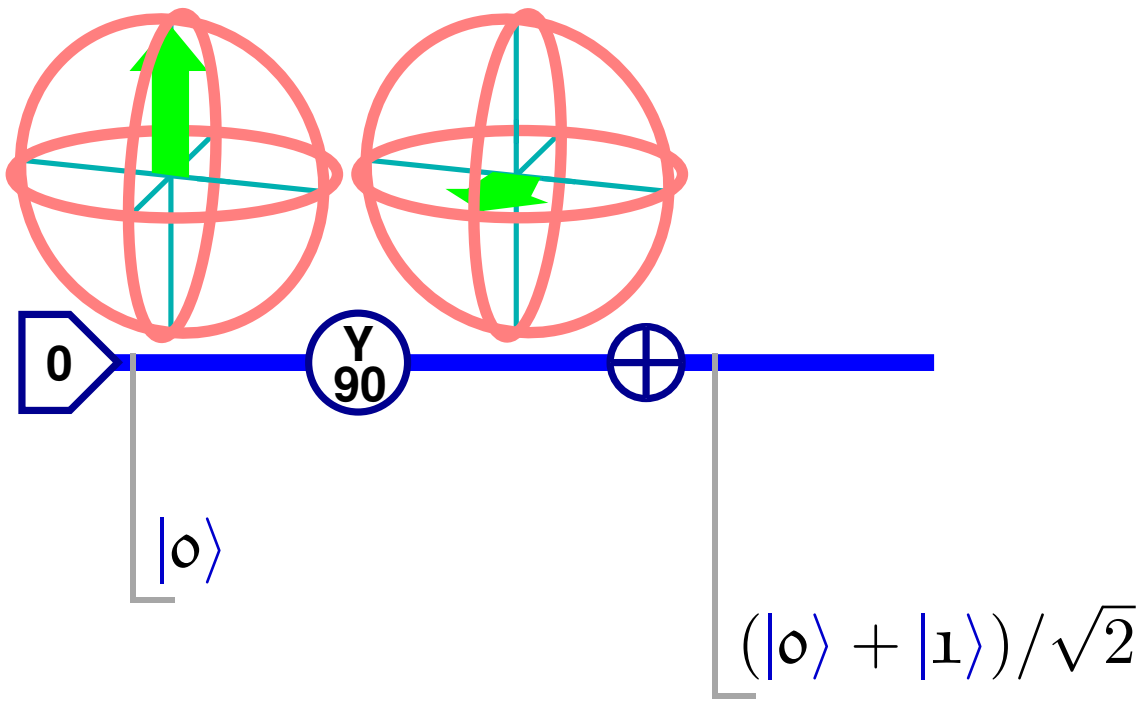

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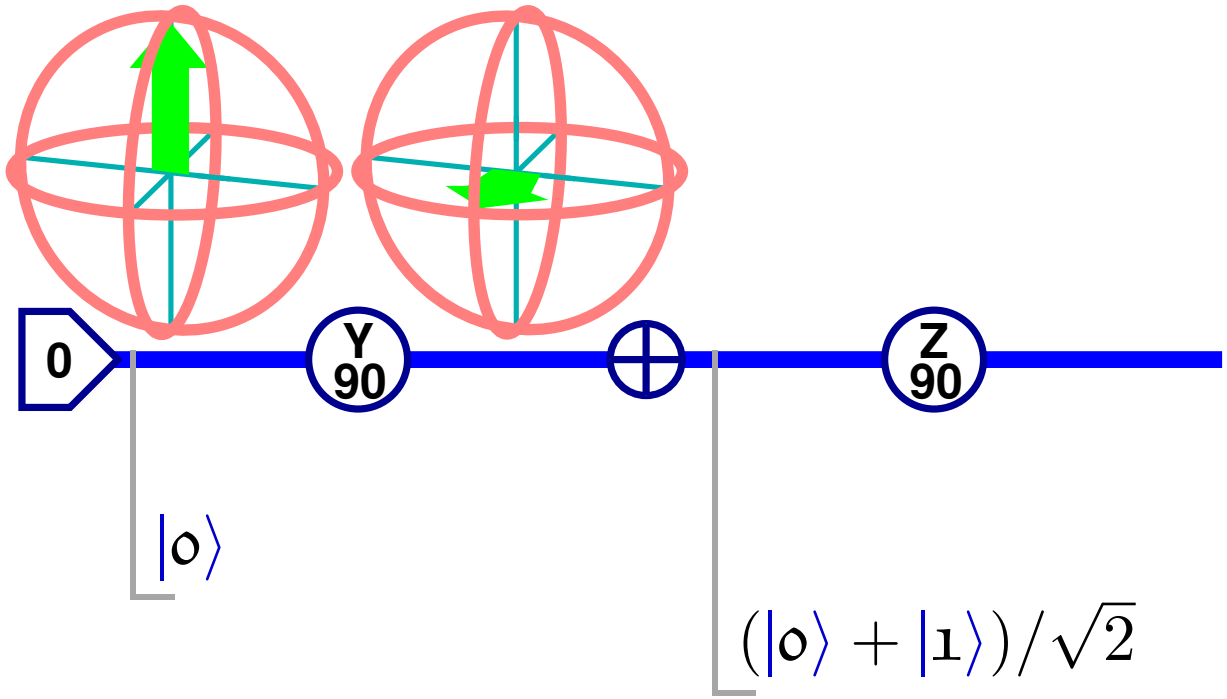

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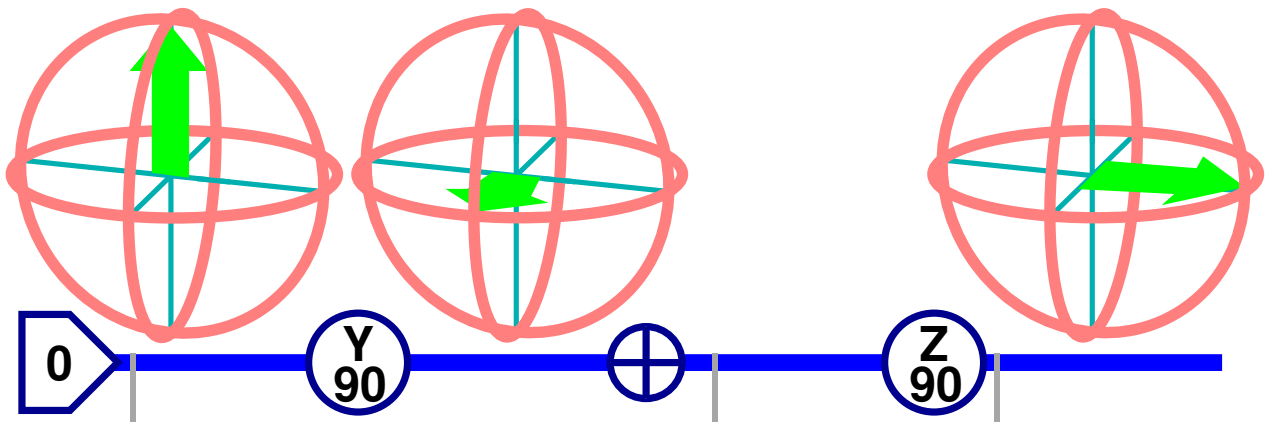

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$(|0\rangle + |1\rangle)/\sqrt{2}$

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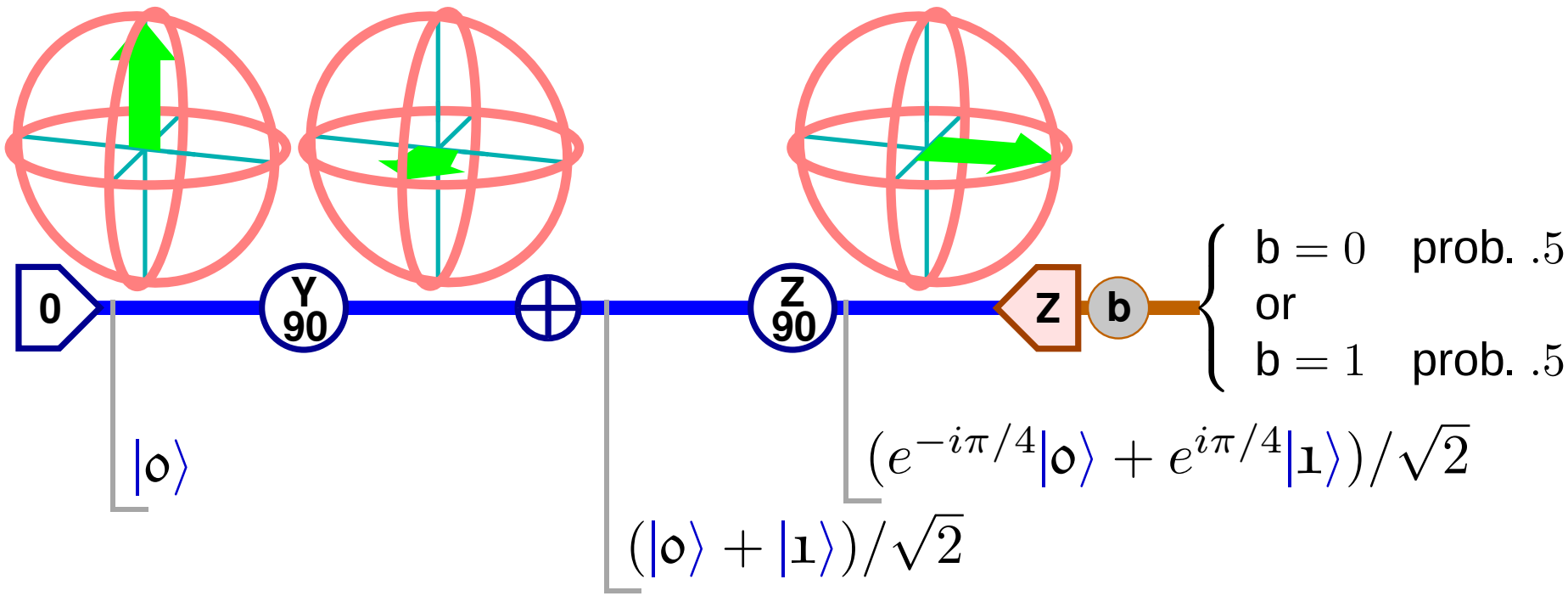

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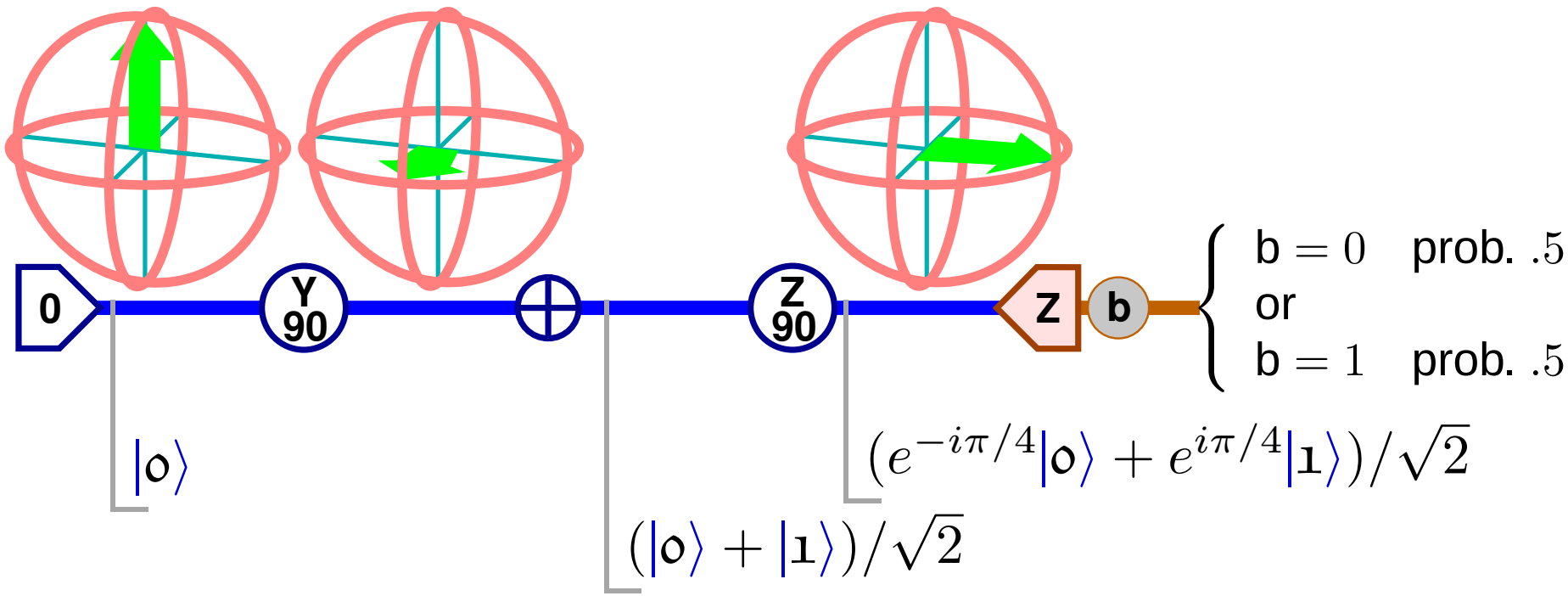

- What rotation should be added before the measurement to guarantee that b $= 1$?

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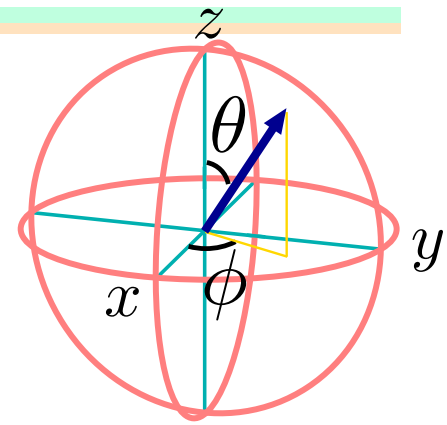
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{(a+d)}{2}\mathbb{1} + \frac{(b+c)}{2}\sigma_x + \frac{i(b-c)}{2}\sigma_y + \frac{(a-d)}{2}\sigma_z$$

States and the Bloch Sphere

$$e^{-i\phi/2}\cos(\theta/2)|\mathtt{0}\rangle + e^{i\phi/2}\sin(\theta/2)|\mathtt{1}\rangle$$



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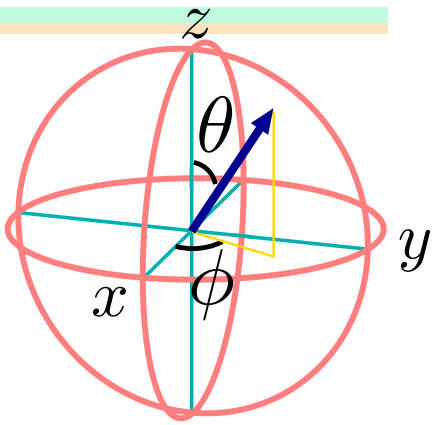

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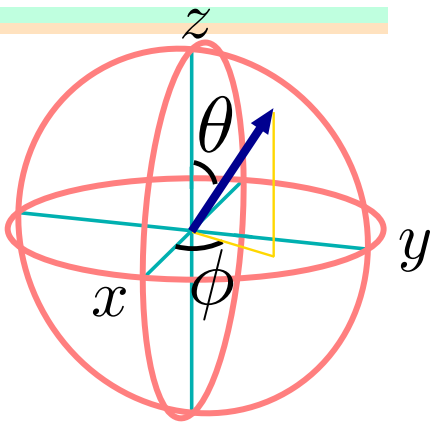

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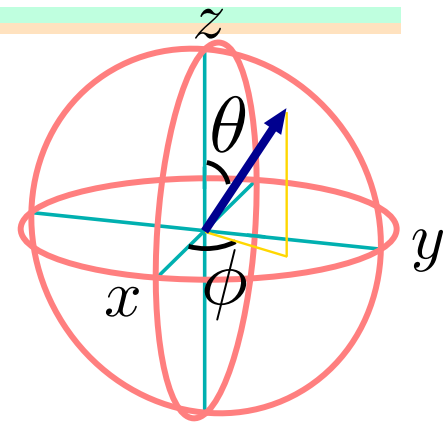

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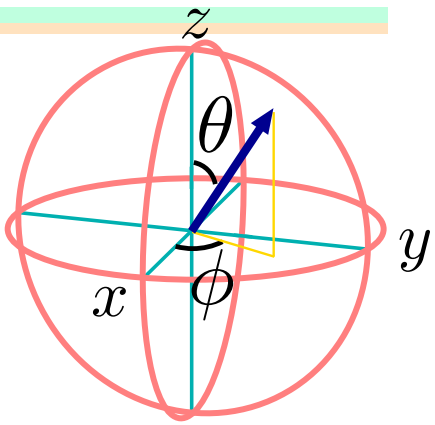

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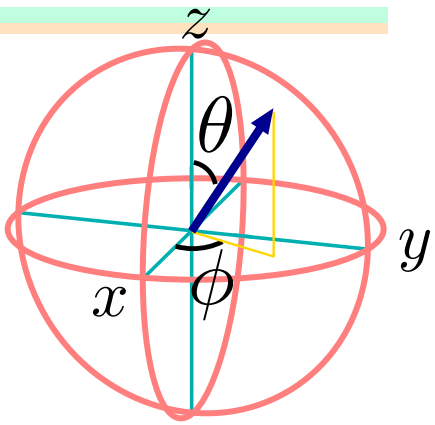

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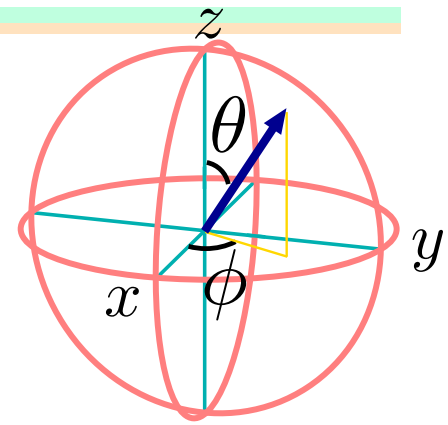

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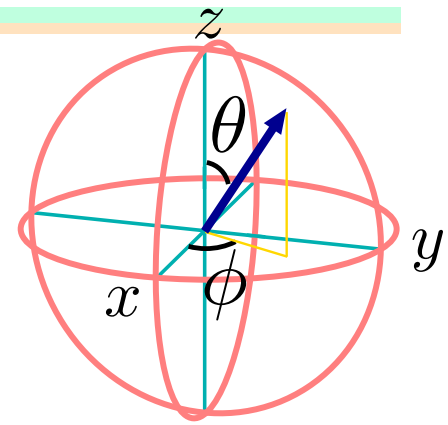

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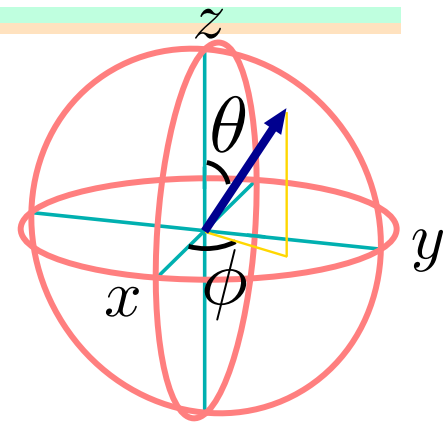

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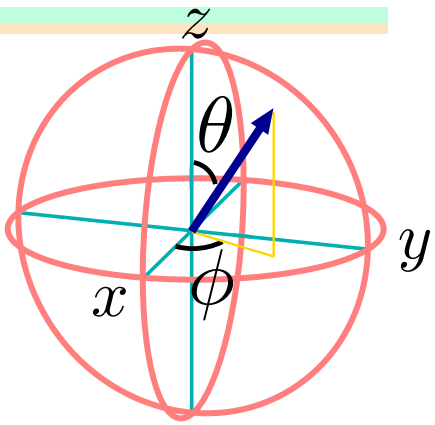

- The *density matrix* ρ for state $\mathbf{x} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ is

$$\rho = \mathbf{x}\mathbf{x}^\dagger = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}(\bar{\alpha}, \bar{\beta}) = \begin{pmatrix}\alpha\bar{\alpha} & \alpha\bar{\beta}\\ \beta\bar{\alpha} & \beta\bar{\beta}\end{pmatrix}$$

  - ρ represents the accessible information about the state. If $\mathbf{y} = e^{i\delta}\mathbf{x}$, then $\mathbf{y}\mathbf{y}^\dagger = e^{i\delta}\mathbf{x}e^{-i\delta}\mathbf{x}^\dagger = \mathbf{x}\mathbf{x}^\dagger$.

- Examples:

$|\mathbf{1}\rangle \rightarrow$

$$\begin{pmatrix}0\\ 1\end{pmatrix}(0,1) = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \tfrac{1}{2}\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)$$

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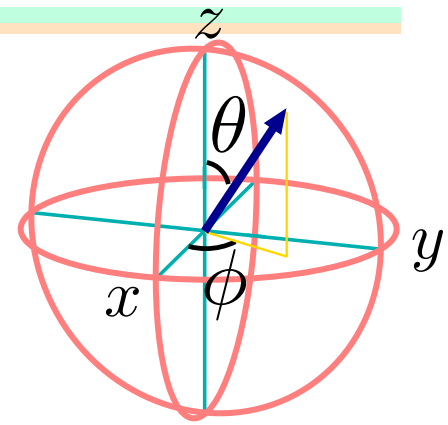

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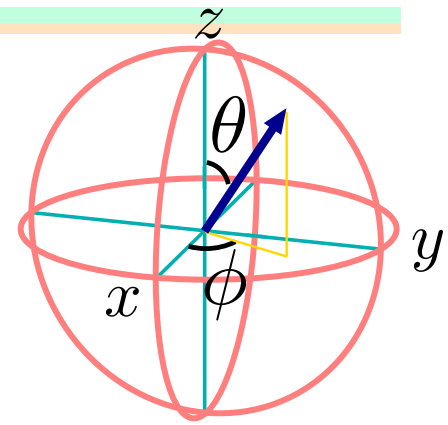

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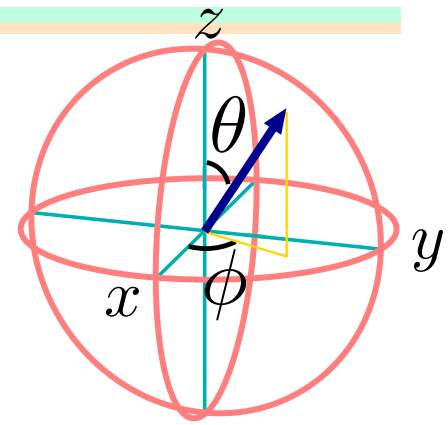

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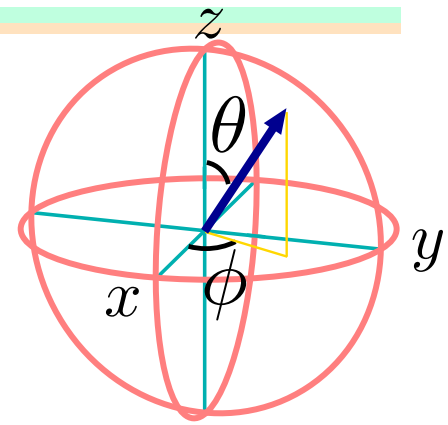

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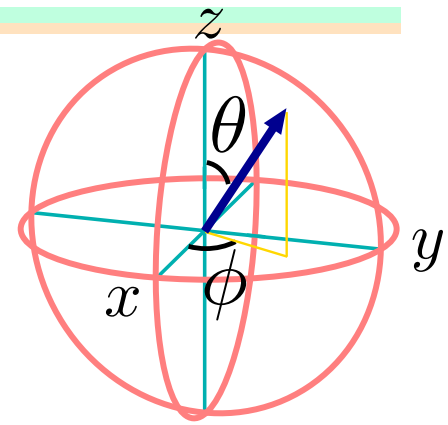

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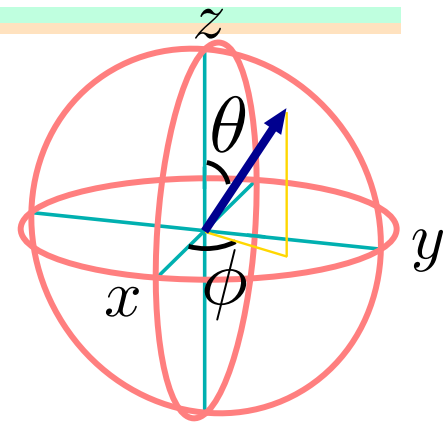

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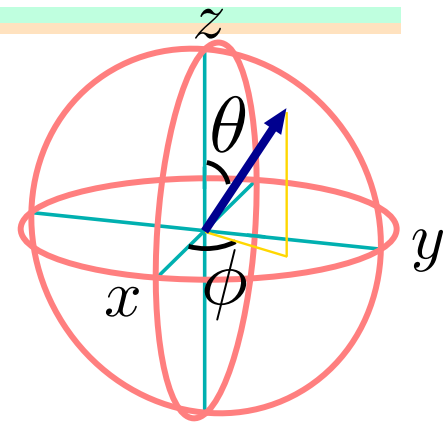

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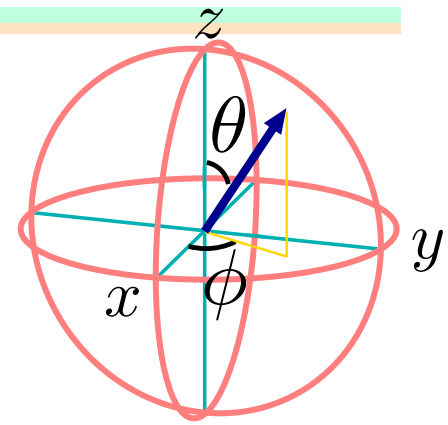

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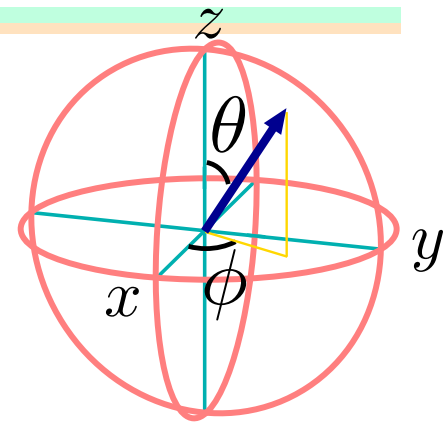

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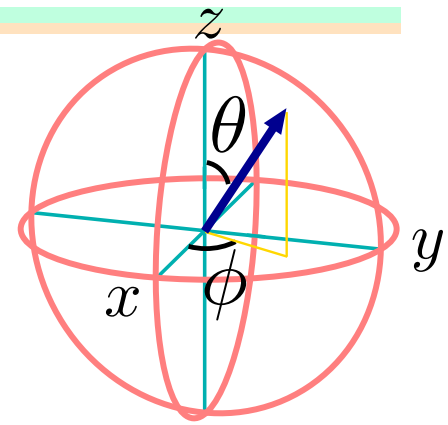

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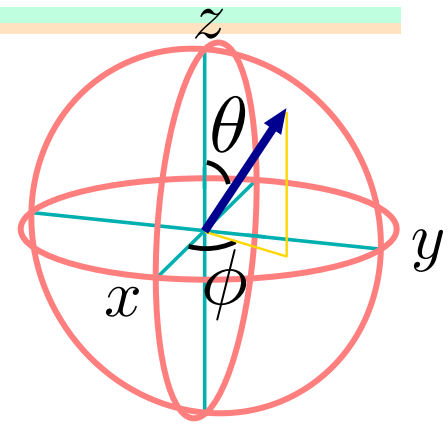

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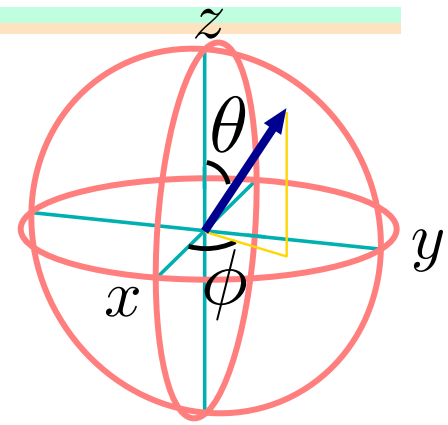

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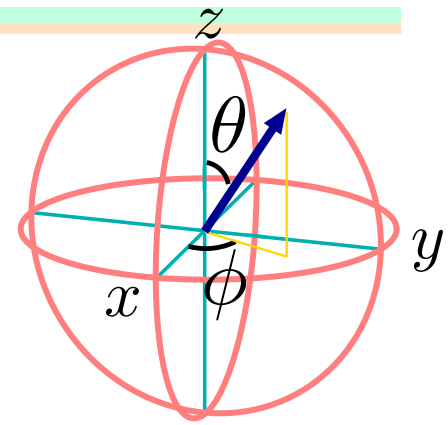

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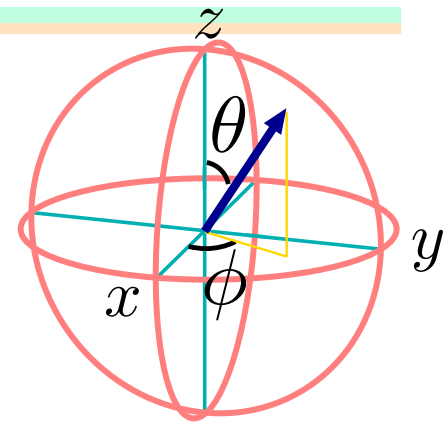

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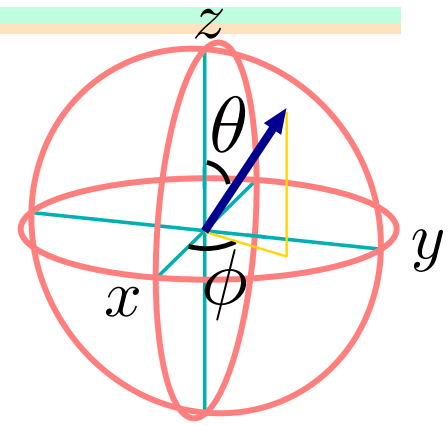

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$$\begin{cases} |0\rangle \rightarrow \frac{1}{2}(\mathbb{1}+\sigma_z) & |1\rangle \rightarrow \frac{1}{2}(\mathbb{1}-\sigma_z) \\ |+\rangle \rightarrow \frac{1}{2}(\mathbb{1}+\sigma_x) & |-\rangle \rightarrow \frac{1}{2}(\mathbb{1}-\sigma_x) \end{cases}$$

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States and the Bloch Sphere

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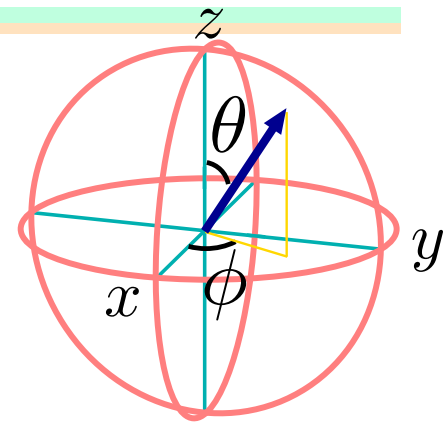

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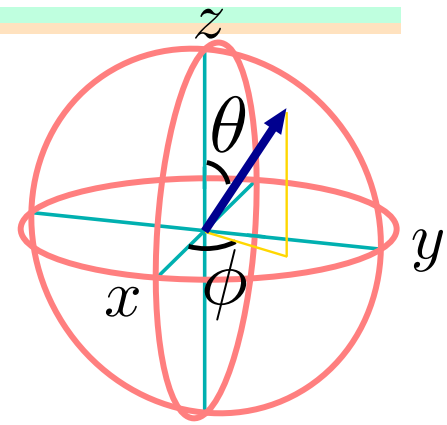

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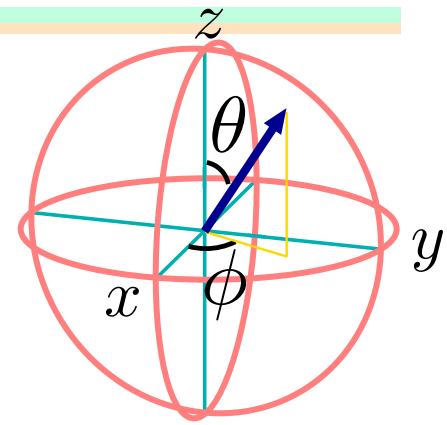

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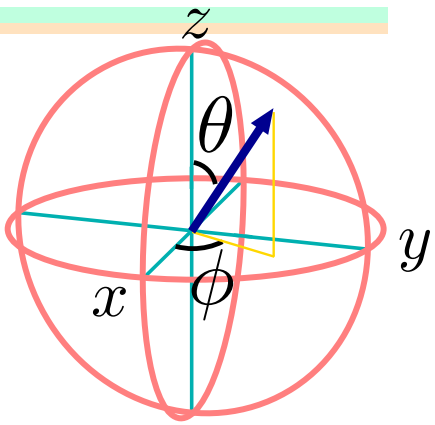

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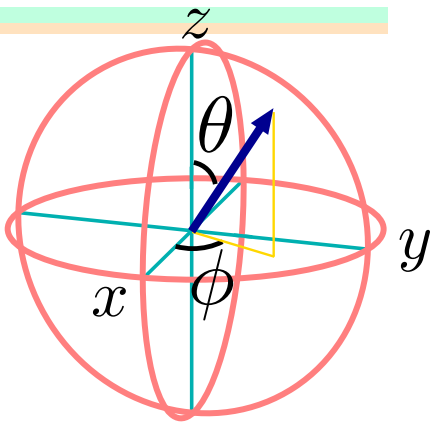

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- $|\psi\rangle$ corresponds to $\hat{u}$ on the Bloch sphere if and only if the density matrix ρ for $|\psi\rangle$ is given by

$$\rho = \tfrac{1}{2}(1 + \hat{u}\cdot\vec{\sigma}) = \tfrac{1}{2}(1 + u_x\sigma_x + u_y\sigma_y + u_z\sigma_z)$$

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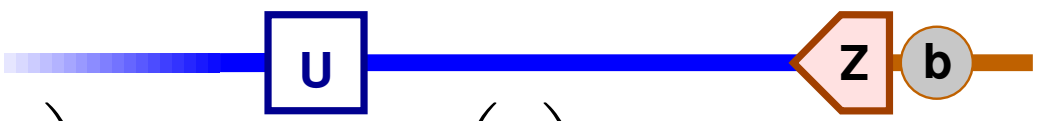

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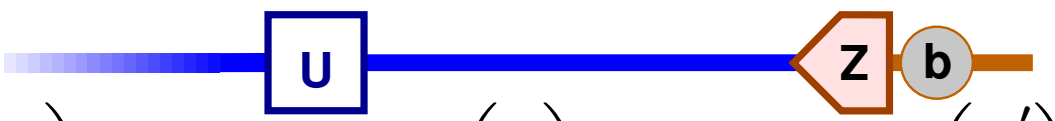

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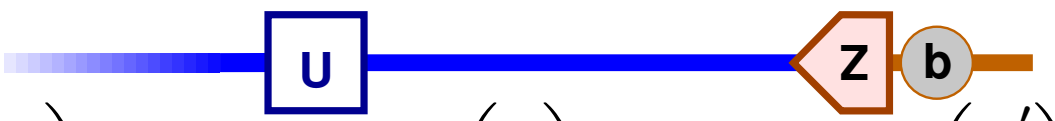

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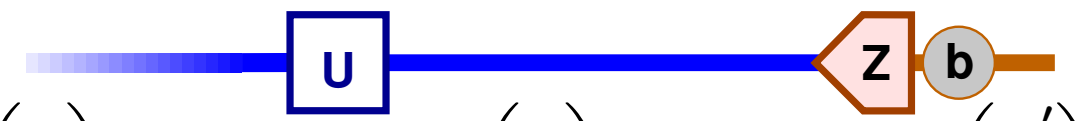

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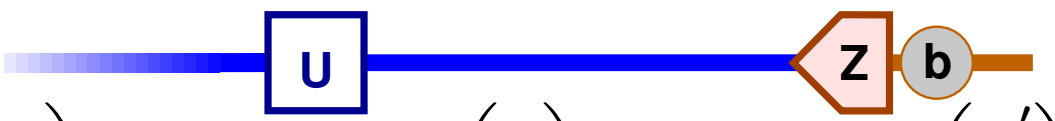

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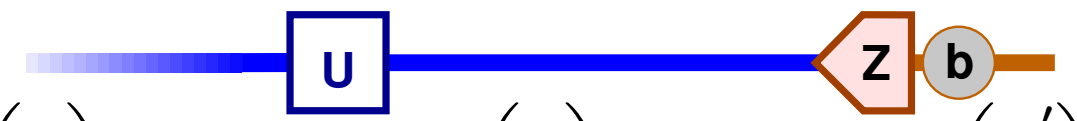

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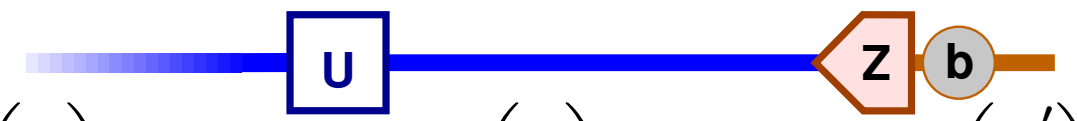

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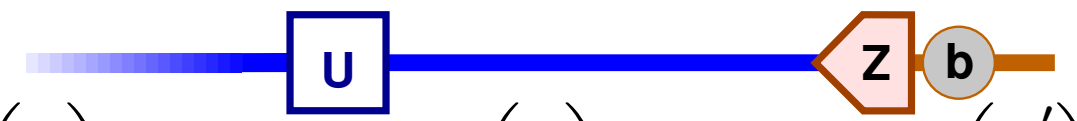

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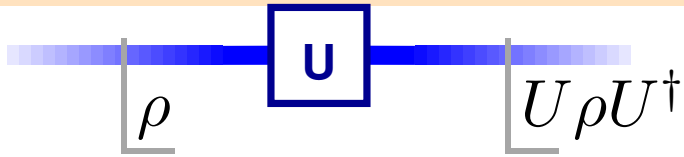
Conjugation



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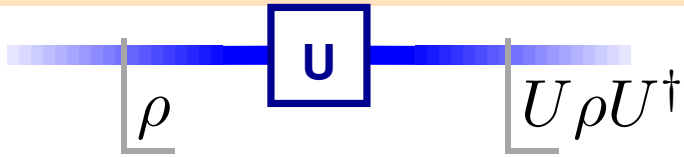

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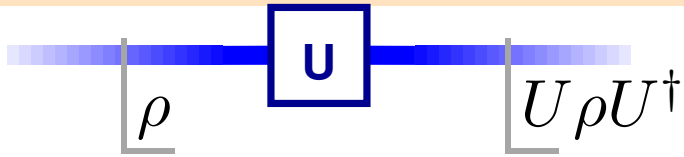

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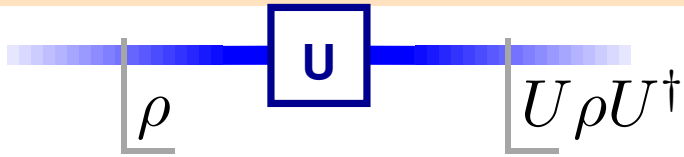

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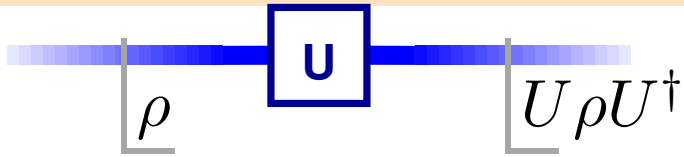

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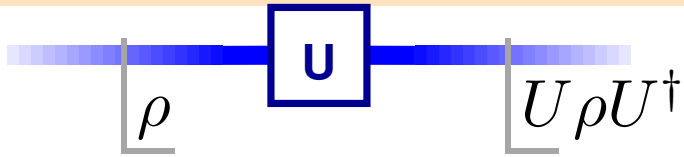

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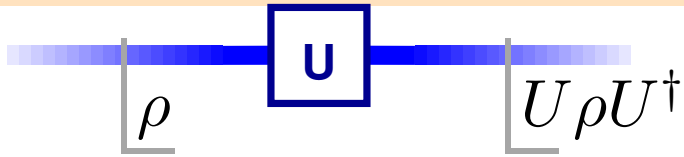

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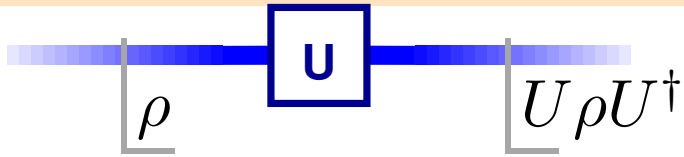

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&= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y
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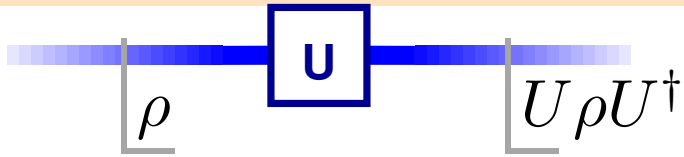

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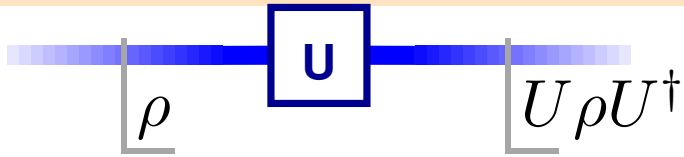

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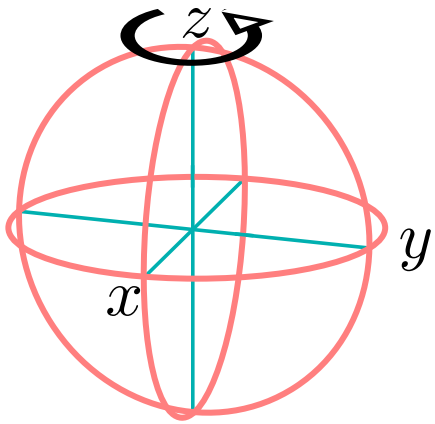

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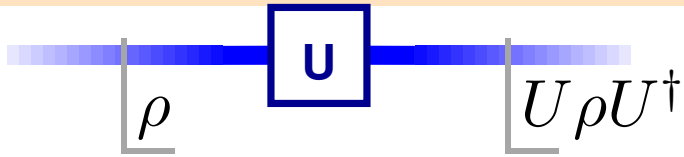

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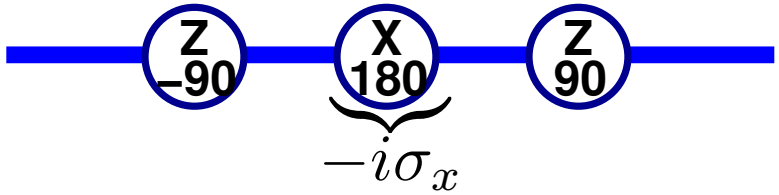

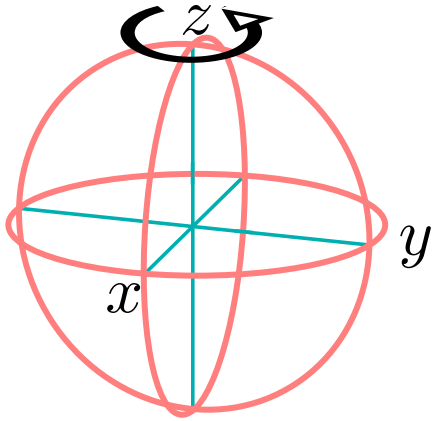

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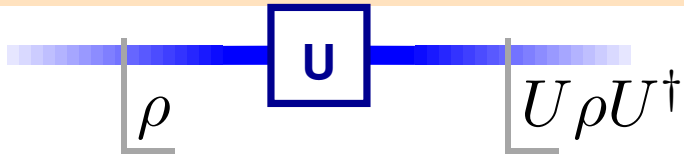

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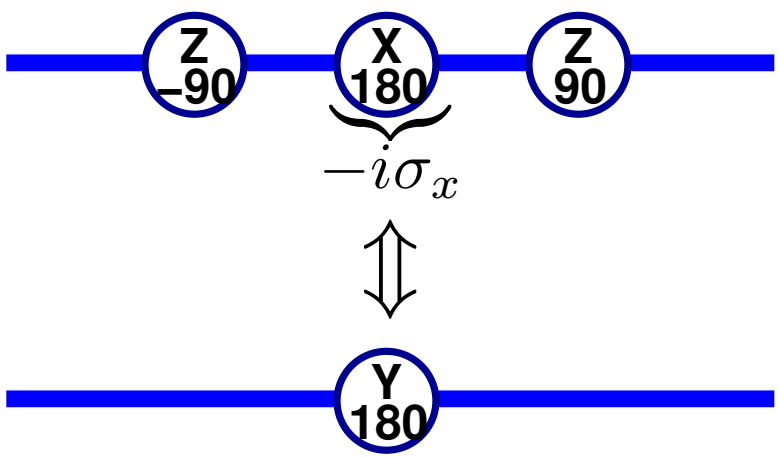

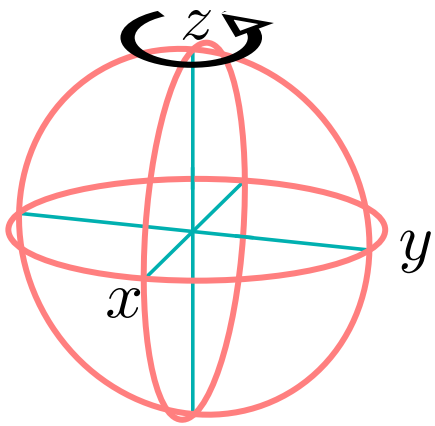

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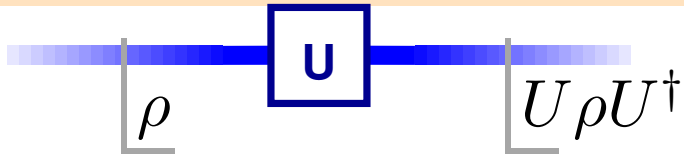

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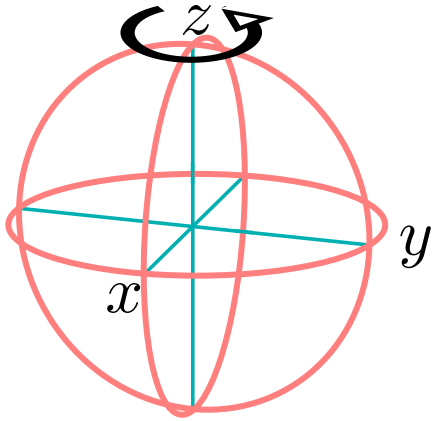

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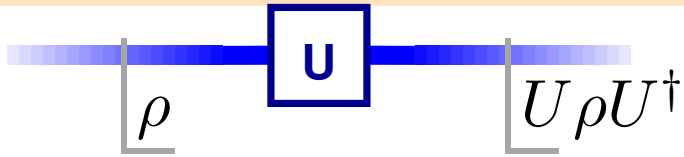

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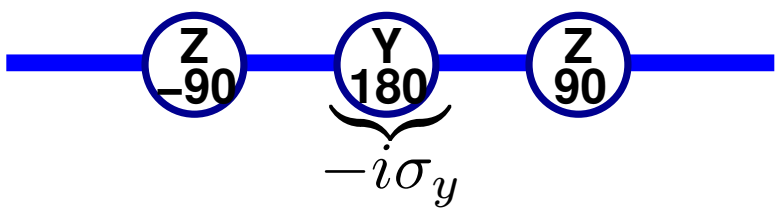

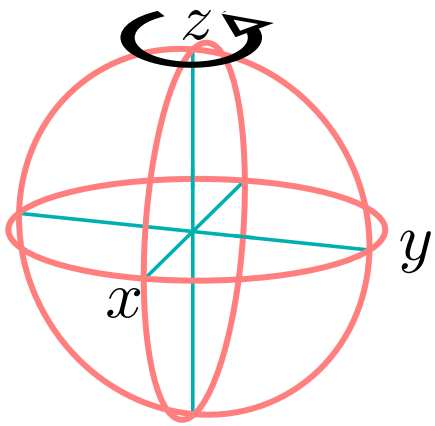

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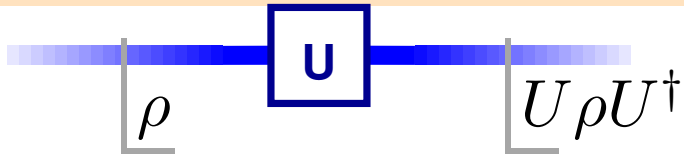

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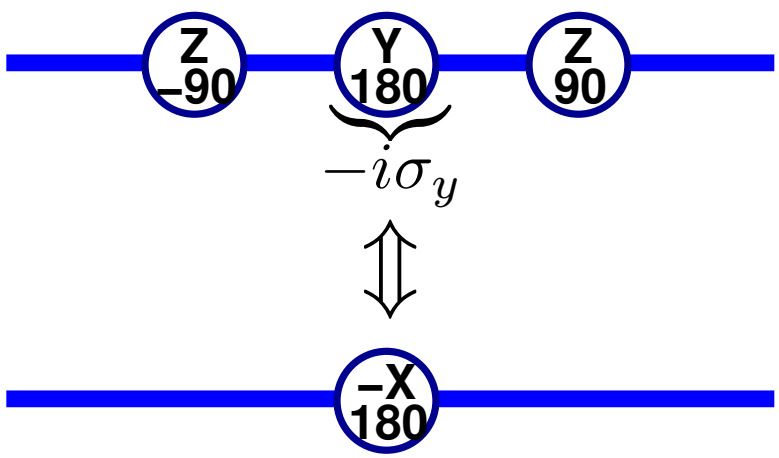
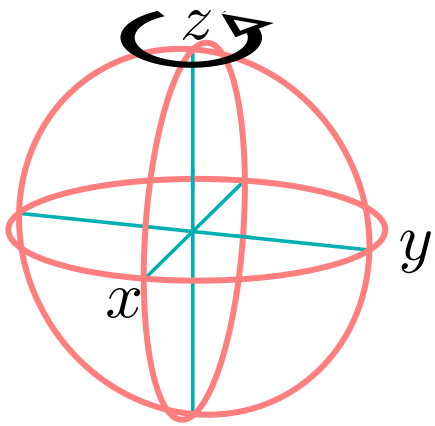
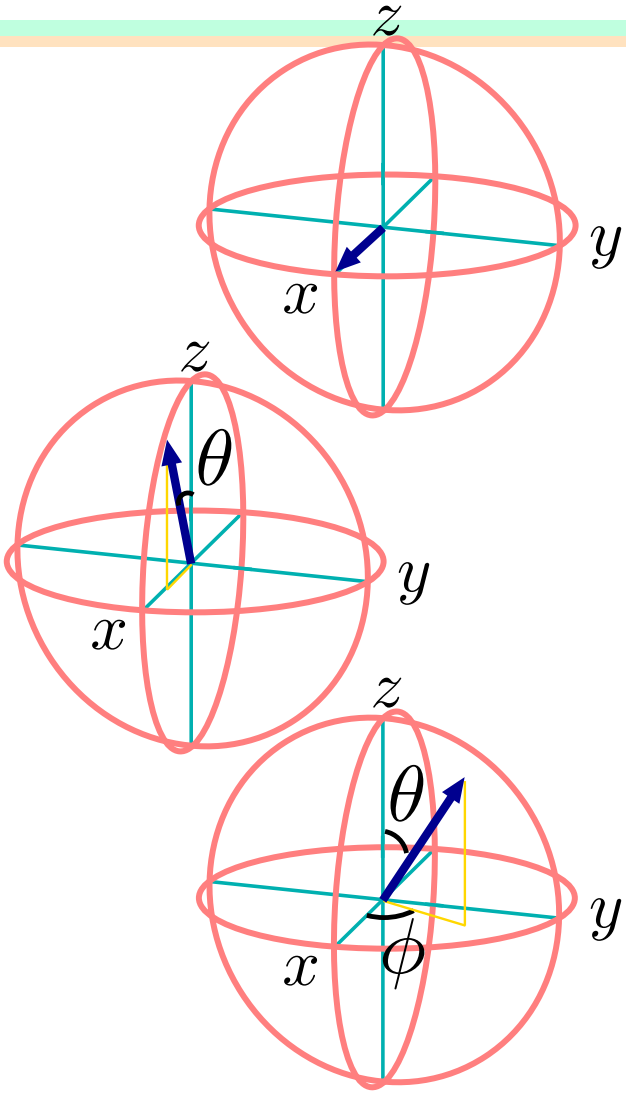




More Conjugation

- 180° rotation around an arbitrary axis.



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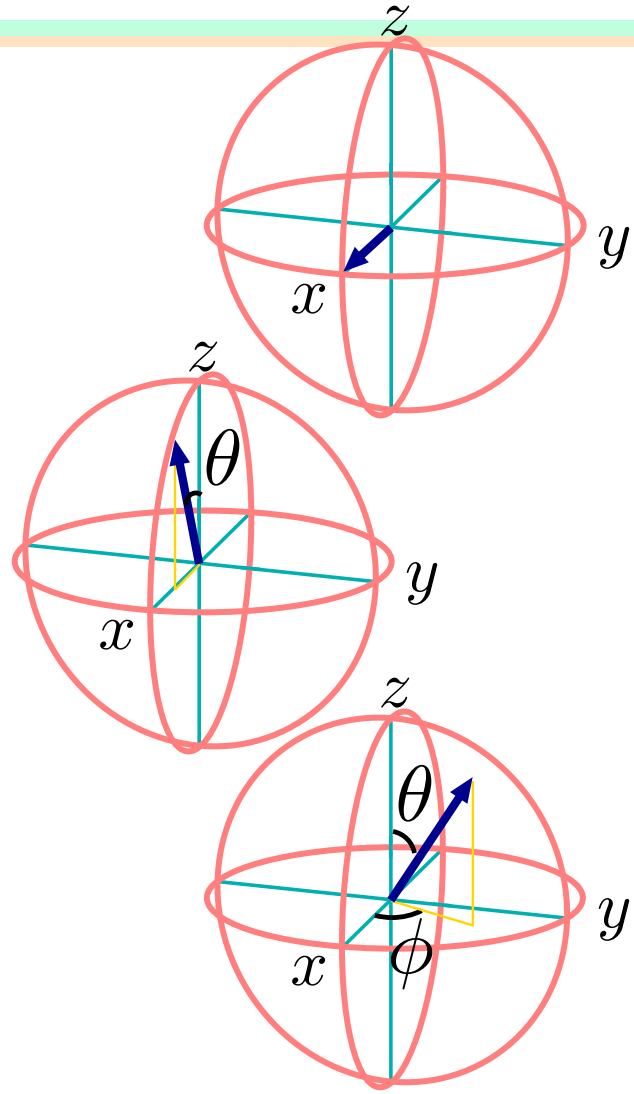

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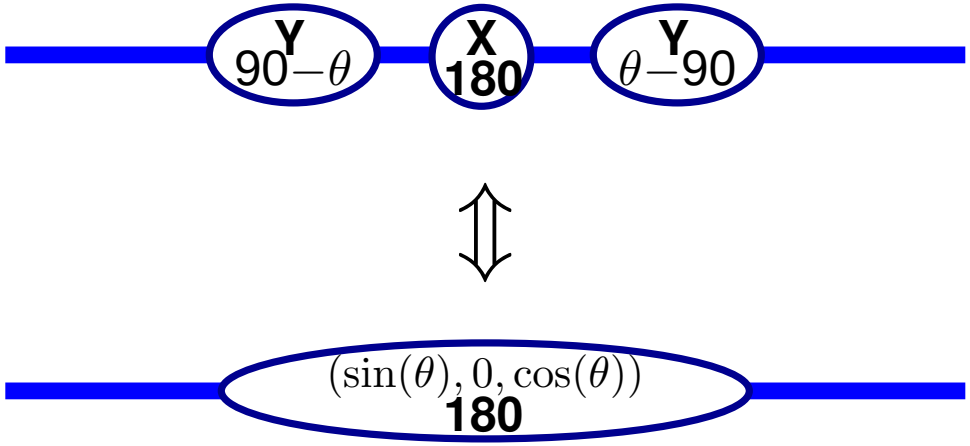

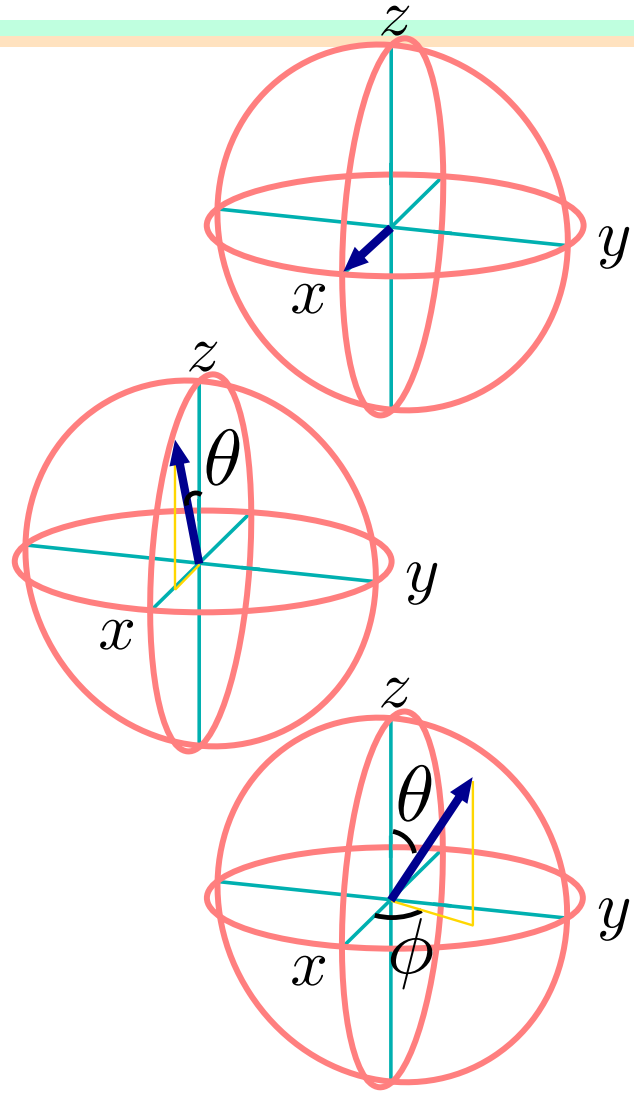

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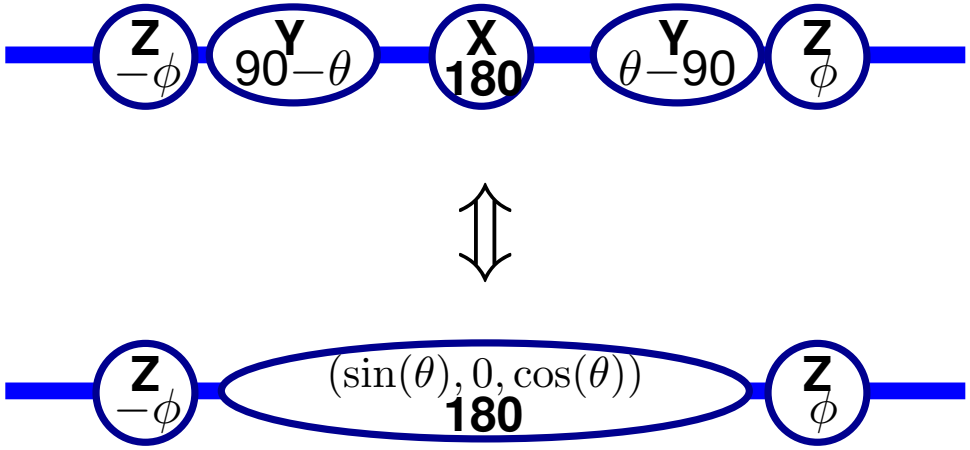

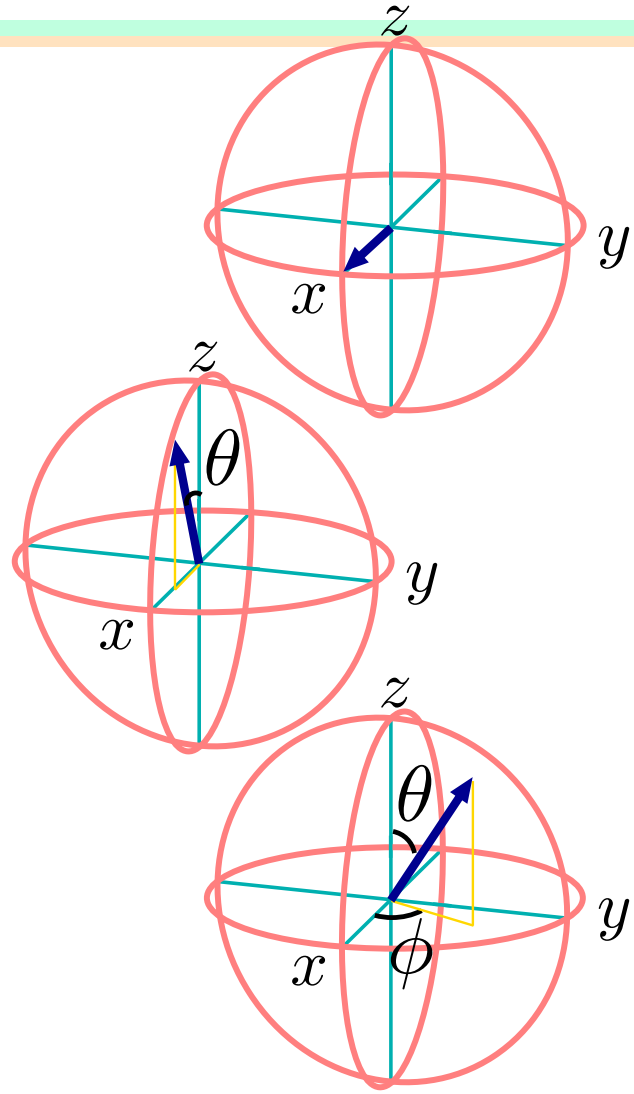

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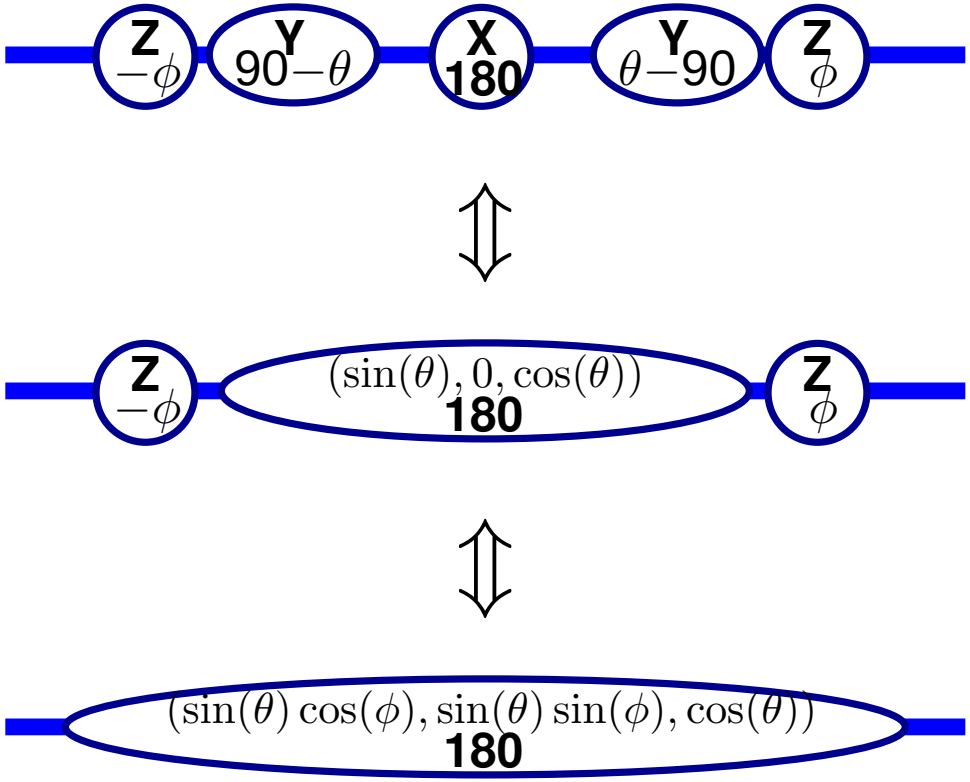

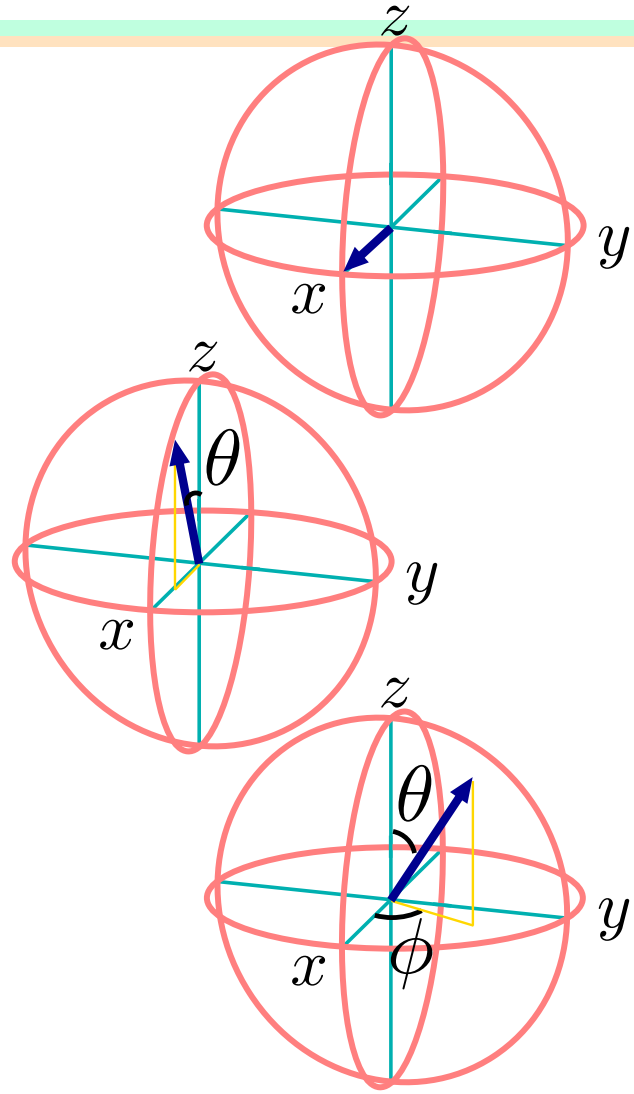

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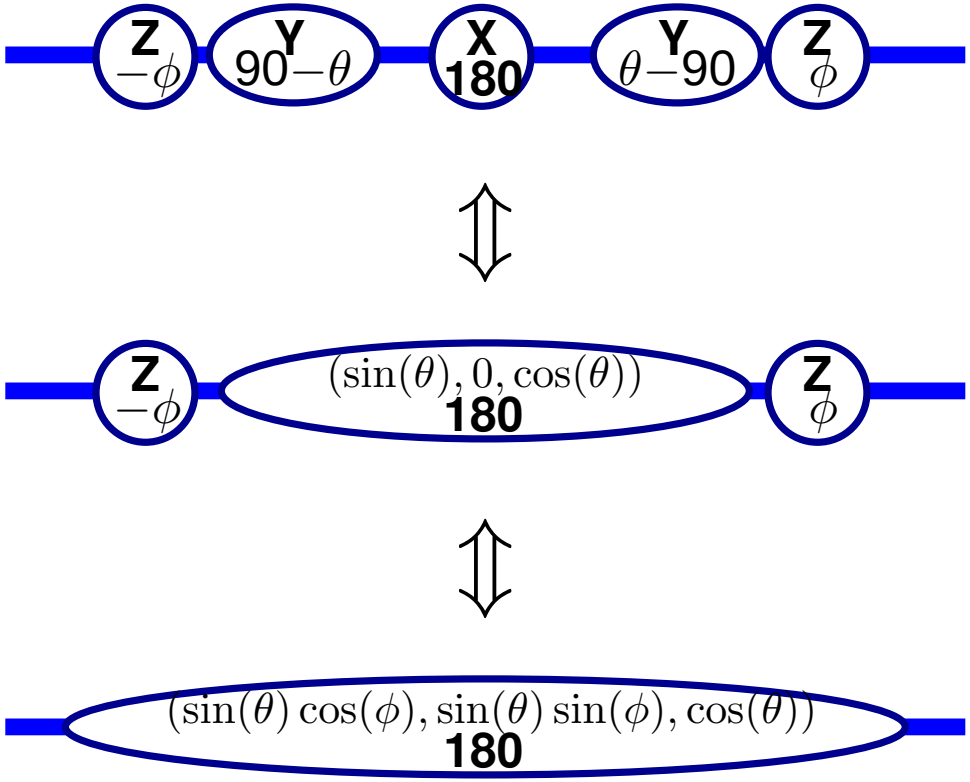

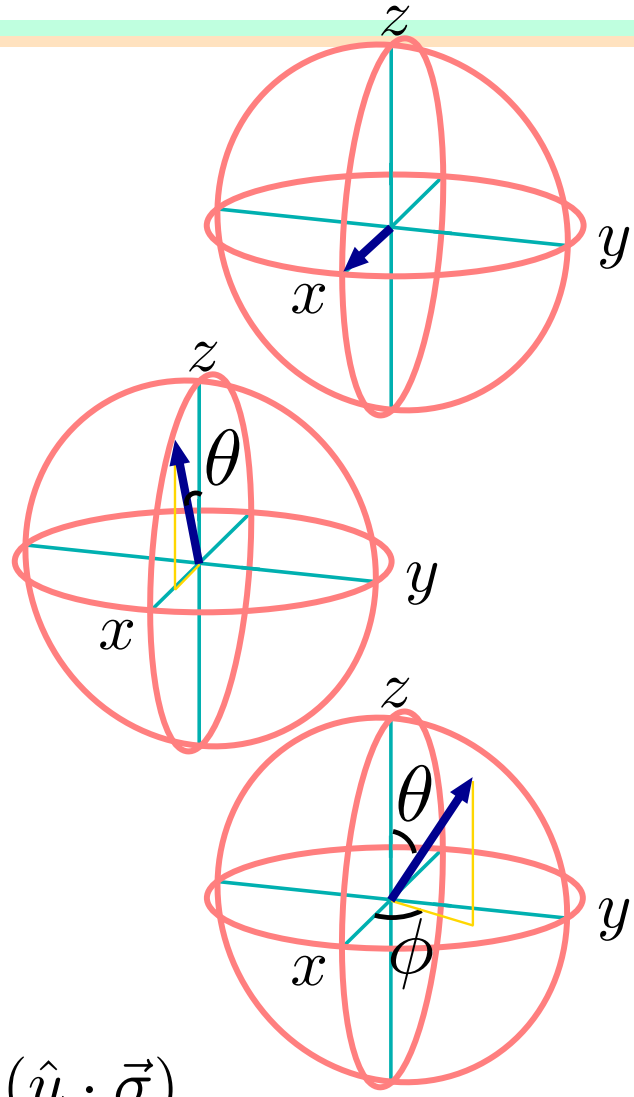

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$$\mathbf{rot}(\hat{u}, \delta) = \cos(\delta/2)\mathbb{1} - i\sin(\delta/2)(\hat{u} \cdot \vec{\sigma}).$$

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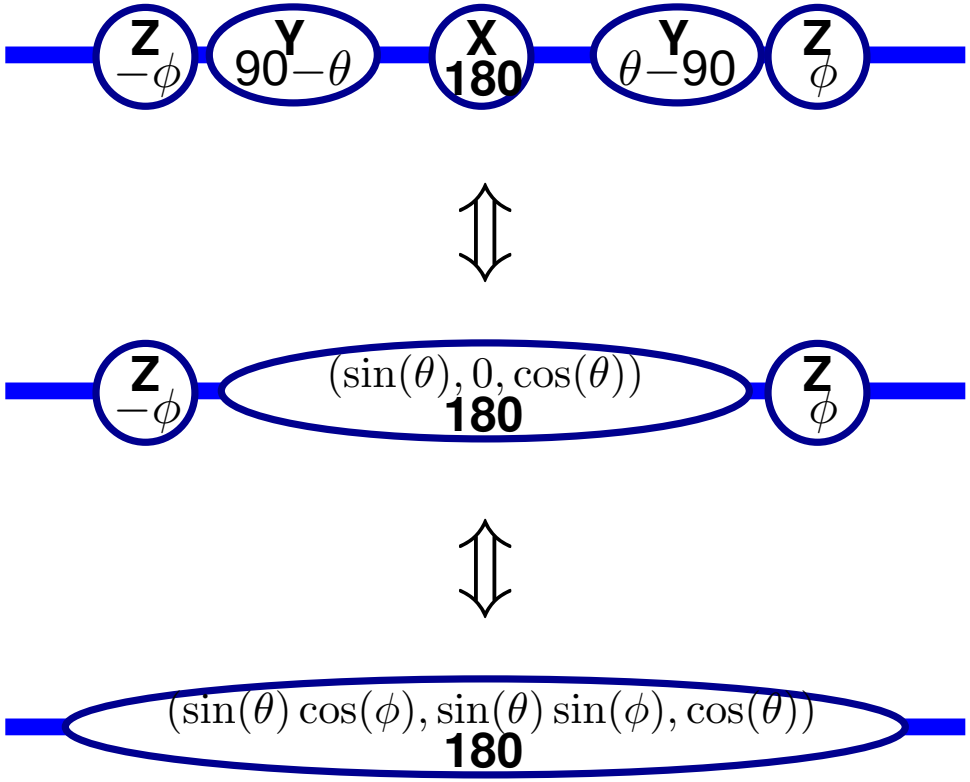

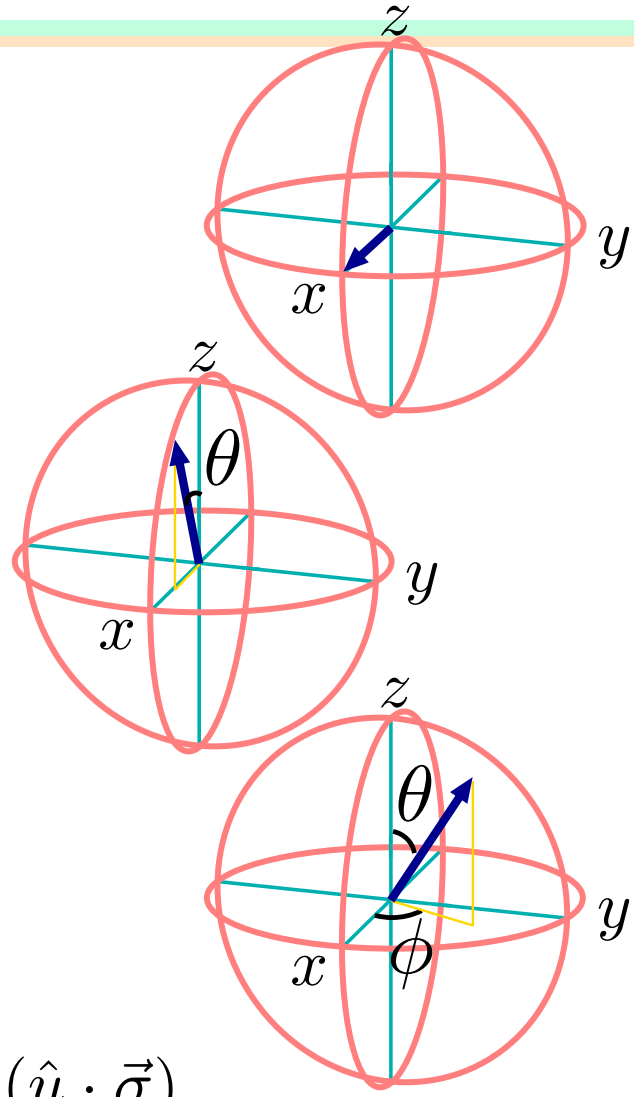

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- Can implement any rotation with 5 major axis rotations.

. . . three are necessary and sufficient.

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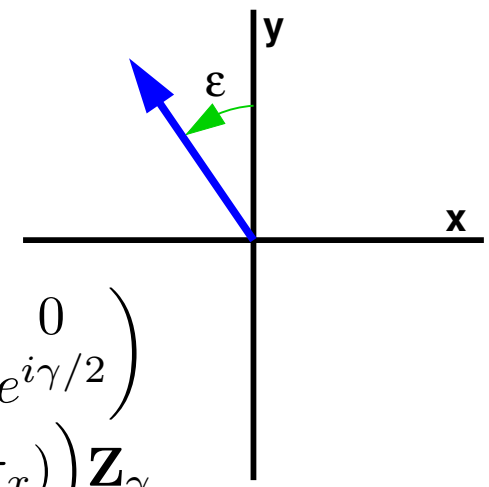

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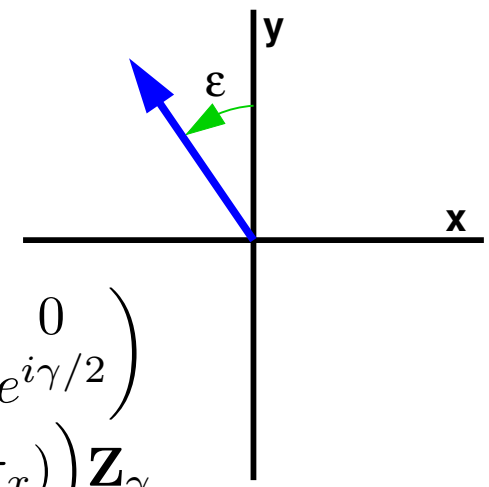

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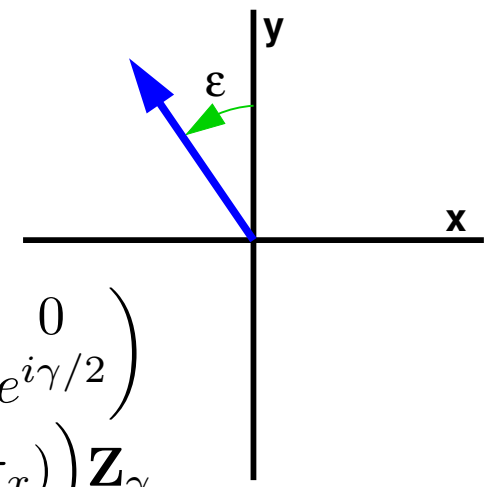

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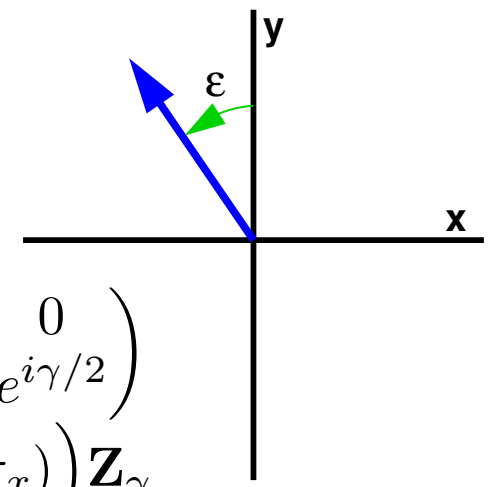

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