IQI 04, Seminar 4

Produced with pdflatex and xfig

- One qubit rotations.
- Universality for one qubit.
- State distinguishability for one qubit.
- Bra-ket notation.
- Application: Quantum cryptography.

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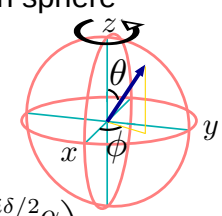
Rotations I

- Rotation gates rotate the state in the Bloch sphere representation.

$$\alpha|0\rangle + \beta|1\rangle \cong e^{-i\phi/2}\cos(\theta/2)|0\rangle + e^{i\phi/2}\sin(\theta/2)|1\rangle$$

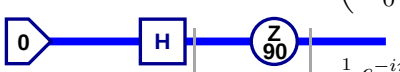


- Z-rotations.

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \xrightarrow{\mathbf{Z}_\delta} \begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix} e^{-i\delta/2}\alpha \\ e^{i\delta/2}\beta\end{pmatrix}$$

- Preparing $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$.

$$\mathbf{Z}_{90^\circ} : \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4}\end{pmatrix}$$

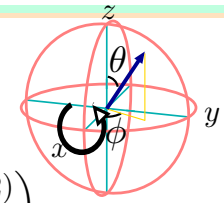


$\frac{1}{\sqrt{2}}e^{-i\pi/4}|0\rangle + \frac{1}{\sqrt{2}}e^{i\pi/4}|1\rangle$
$= \frac{1}{\sqrt{2}}e^{-i\pi/4}(|0\rangle + i|1\rangle)$

$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

Rotations II

$$\alpha|0\rangle + \beta|1\rangle \cong e^{-i\phi/2}\cos(\theta/2)|0\rangle + e^{i\phi/2}\sin(\theta/2)|1\rangle$$



- Y-rotations.

$$\mathbf{Y}_\delta : \begin{pmatrix} \cos(\delta/2) & -\sin(\delta/2) \\ \sin(\delta/2) & \cos(\delta/2)\end{pmatrix}$$

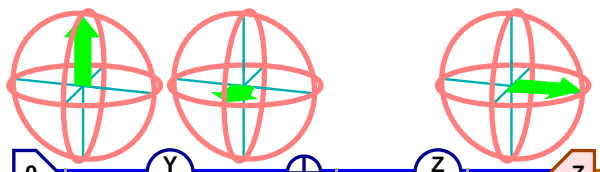
$$\left.\begin{matrix}|0\rangle \\ |1\rangle\end{matrix}\right\} \text{—} \boxed{Y_\delta} \text{—} \left\{\begin{matrix}\cos(\delta/2)|0\rangle + \sin(\delta/2)|1\rangle \\ -\sin(\delta/2)|0\rangle + \cos(\delta/2)|1\rangle\end{matrix}\right.$$

- X-rotations.

$$\mathbf{X}_\delta : \begin{pmatrix} \cos(\delta/2) & -i\sin(\delta/2) \\ -i\sin(\delta/2) & \cos(\delta/2)\end{pmatrix}$$

$$\left.\begin{matrix}|0\rangle \\ |1\rangle\end{matrix}\right\} \text{—} \boxed{X_\delta} \text{—} \left\{\begin{matrix}\cos(\delta/2)|0\rangle - i\sin(\delta/2)|1\rangle \\ -i\sin(\delta/2)|0\rangle + \cos(\delta/2)|1\rangle\end{matrix}\right.$$

A One-Qubit Network



$|0\rangle$

$(|0\rangle + |1\rangle)/\sqrt{2}$

$(|0\rangle + i|1\rangle)/\sqrt{2}$

$\left\{\begin{matrix} b = 0 & \text{prob. } .5 \\ \text{or} & \\ b = 1 & \text{prob. } .5\end{matrix}\right.$

- What rotation should be added before the measurement to guarantee that $b = 1$?

The Pauli Matrices

- Define the Pauli matrices by

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Rotations in terms of Pauli matrices.

$$\mathbf{X}_\delta = \begin{pmatrix} \cos(\delta/2) & -i\sin(\delta/2) \\ -i\sin(\delta/2) & \cos(\delta/2) \end{pmatrix} = \cos(\delta/2)\mathbb{1} - i\sin(\delta/2)\sigma_x$$

$$\mathbf{Y}_\delta = \begin{pmatrix} \cos(\delta/2) & -\sin(\delta/2) \\ \sin(\delta/2) & \cos(\delta/2) \end{pmatrix} = \cos(\delta/2)\mathbb{1} - i\sin(\delta/2)\sigma_y$$

$$\mathbf{Z}_\delta = \begin{pmatrix} \cos(\delta/2) - i\sin(\delta/2) & 0 \\ 0 & \cos(\delta/2) + i\sin(\delta/2) \end{pmatrix} = \cos(\delta/2)\mathbb{1} - i\sin(\delta/2)\sigma_z$$

- Relationship to the first set of gates.
  - $\sigma_z = \mathbf{not} = i\mathbf{Z}_{180^\circ}$.
  - $\sigma_x = \mathbf{sgn} = i\mathbf{X}_{180^\circ}$.
  - $\sigma_y = i\mathbf{sgn.not} = i\mathbf{Y}_{180^\circ}$.

Properties of Pauli Matrices

- Define the Pauli matrices by

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The Hermitian transpose of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^\dagger = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$.
- $\sigma_u^\dagger = \sigma_u$, that is, the Pauli matrices are *Hermitian*.
- $\sigma_u^2 = \mathbb{1}$.
- The Pauli matrices *anticommute*:

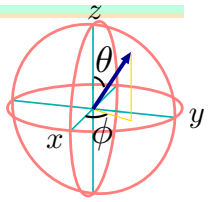
$$\sigma_x\sigma_y = -\sigma_y\sigma_x, \ \sigma_y\sigma_z = -\sigma_z\sigma_y, \ \sigma_z\sigma_x = -\sigma_x\sigma_z.$$

- The Pauli matrices form an operator basis:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{(a+d)}{2}\mathbb{1} + \frac{(b+c)}{2}\sigma_x + \frac{i(b-c)}{2}\sigma_y + \frac{(a-d)}{2}\sigma_z$$

States and the Bloch Sphere

$$e^{-i\phi/2}\cos(\theta/2)|\mathbf{0}\rangle + e^{i\phi/2}\sin(\theta/2)|\mathbf{1}\rangle$$



- The *density matrix* ρ for state $\mathbf{x} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is

$$\rho = \mathbf{x}\mathbf{x}^\dagger = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (\bar{\alpha}, \bar{\beta}) = \begin{pmatrix} \alpha\bar{\alpha} & \alpha\bar{\beta} \\ \beta\bar{\alpha} & \beta\bar{\beta} \end{pmatrix}$$

  - ρ represents the accessible information about the state. If $\mathbf{y} = e^{i\delta}\mathbf{x}$, then $\mathbf{y}\mathbf{y}^\dagger = e^{i\delta}\mathbf{x}e^{-i\delta}\mathbf{x}^\dagger = \mathbf{x}\mathbf{x}^\dagger$.

- Examples:

$$\left\{ \begin{array}{lclclcl} |\mathbf{0}\rangle & \to & \frac{1}{2}(\mathbb{1}+\sigma_z) & & |\mathbf{1}\rangle & \to & \frac{1}{2}(\mathbb{1}-\sigma_z) \\ |+\rangle & \to & \frac{1}{2}(\mathbb{1}+\sigma_x) & & |-\rangle & \to & \frac{1}{2}(\mathbb{1}-\sigma_x) \\ |+i\rangle & \to & \frac{1}{2}(\mathbb{1}+\sigma_y) & & |-i\rangle & \to & \frac{1}{2}(\mathbb{1}-\sigma_y) \end{array} \right.$$

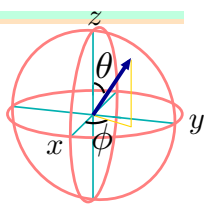
$$|-i\rangle = \frac{1}{\sqrt{2}}(|\mathbf{0}\rangle - i|\mathbf{1}\rangle) \to$$

$$\frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}(1, i) = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right)$$

$$= \frac{1}{2}(\mathbb{1} - \sigma_y)$$

States and the Bloch Sphere

$$e^{-i\phi/2}\cos(\theta/2)|\mathbf{0}\rangle + e^{i\phi/2}\sin(\theta/2)|\mathbf{1}\rangle$$



- The *density matrix* ρ for state $\mathbf{x} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is

$$\rho = \mathbf{x}\mathbf{x}^\dagger = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (\bar{\alpha}, \bar{\beta}) = \begin{pmatrix} \alpha\bar{\alpha} & \alpha\bar{\beta} \\ \beta\bar{\alpha} & \beta\bar{\beta} \end{pmatrix}$$

  - ρ represents the accessible information about the state. If $\mathbf{y} = e^{i\delta}\mathbf{x}$, then $\mathbf{y}\mathbf{y}^\dagger = e^{i\delta}\mathbf{x}e^{-i\delta}\mathbf{x}^\dagger = \mathbf{x}\mathbf{x}^\dagger$.

- $|\psi\rangle$ corresponds to $\hat{u}$ on the Bloch sphere if and only if the density matrix ρ for $|\psi\rangle$ is given by

$$\rho = \frac{1}{2}(1 + \hat{u}\cdot\vec{\sigma}) = \frac{1}{2}(1 + u_x\sigma_x + u_y\sigma_y + u_z\sigma_z)$$

The Density Matrix

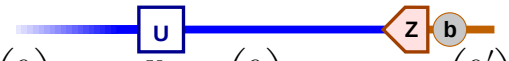
- Let ρ be the density matrix for $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \alpha\bar{\alpha} & \alpha\bar{\beta} \\ \beta\bar{\alpha} & \beta\bar{\beta} \end{pmatrix}.$$

  - ρ_{11} is the probability of measuring 0.
  - ρ_{22} is the probability of measuring 1.

$$\mathrm{tr}(\rho) = \rho_{11} + \rho_{22} = 1$$

- Accessible information. All information about $|\psi\rangle$ is obtained by applying operations and measuring.



$$|\psi\rangle \leftrightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \xrightarrow{U} U\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \doteq \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} \rightarrow \begin{cases} 0 & \text{prob. } \alpha'\bar{\alpha}' \\ 1 & \text{prob. } \beta'\bar{\beta}' \end{cases}$$

$$\leftrightarrow \rho \xrightarrow{U} U\rho U^\dagger \doteq \rho' \rightarrow \begin{cases} 0 & \text{prob. } \rho'_{11} \\ 1 & \text{prob. } \rho'_{22} \end{cases}$$

  - All accessible information is in ρ

. . . and ρ can be determined from accessible information.

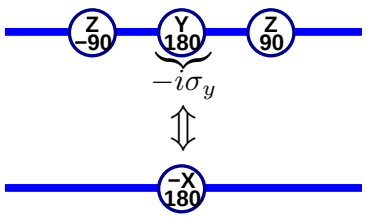
Conjugation

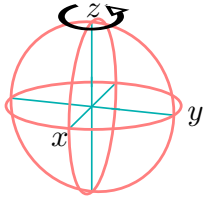
- Applying U *conjugates* density matrix ρ by U.
- Example: $U = \mathbf{Z}_{90^\circ} = \frac{1}{\sqrt{2}}(\mathbb{1} - i\sigma_z) = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$,

$$\rho = \tfrac{1}{2}(\mathbb{1} + \sigma_x).$$

$$\begin{aligned} U\rho U^\dagger &= \tfrac{1}{2}(U\mathbb{1}U^\dagger + U\sigma_x U^\dagger) \\ &= \tfrac{1}{2}(\mathbb{1} + U\sigma_x U^\dagger) = \tfrac{1}{2}(\mathbb{1} + \sigma_y) \end{aligned}$$

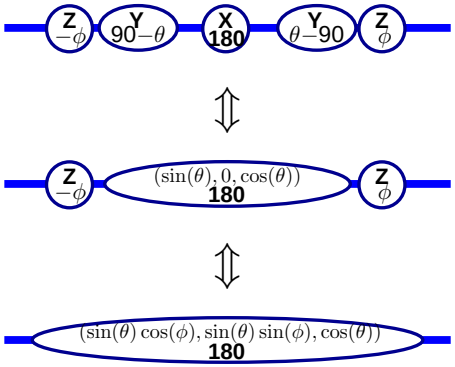
- Simplify:

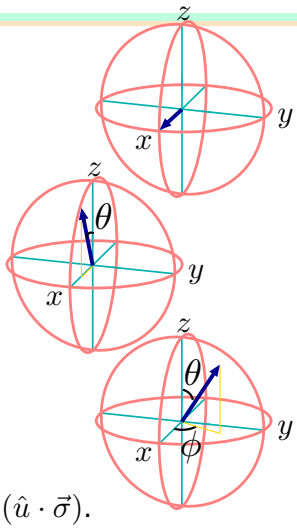




More Conjugation

- 180° rotation around an arbitrary axis.





- Rotation with axis $\hat{u}$ by angle δ:

$$\mathbf{rot}(\hat{u}, \delta) = \cos(\delta/2)\mathbb{1} - i\sin(\delta/2)(\hat{u} \cdot \vec{\sigma}).$$

- Can implement any rotation with 5 major axis rotations.

. . . three are necessary and sufficient.

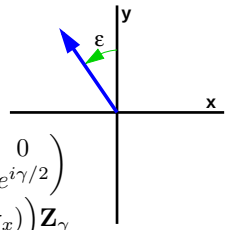
Universality for One Qubit

- Every unitary 2×2 matrix is proportional to a rotation.

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}.$$

1. May take u_{11} to be real.
2. $U^\dagger U = \mathbb{1}$ implies $|u_{11}|^2 + |u_{21}|^2 = 1$: Write $U = \begin{pmatrix} \cos(\delta/2) & u_{12} \\ e^{i\epsilon}\sin(\delta/2) & u_{22} \end{pmatrix}$
3. $(\cos(\delta/2), e^{-i\epsilon}\sin(\delta/2)) \begin{pmatrix} u_{12} \\ u_{22} \end{pmatrix} = 0.$

$$\begin{aligned} U &= \begin{pmatrix} \cos(\delta/2) & -e^{i\gamma}e^{-i\epsilon}\sin(\delta/2) \\ e^{i\epsilon}\sin(\delta/2) & e^{i\gamma}\cos(\delta/2) \end{pmatrix} \\ &= e^{i\gamma/2}\begin{pmatrix} e^{-i\gamma/2}\cos(\delta/2) & -e^{i\gamma/2}e^{-i\epsilon}\sin(\delta/2) \\ e^{-i\gamma/2}e^{i\epsilon}\sin(\delta/2) & e^{i\gamma/2}\cos(\delta/2) \end{pmatrix} \\ &= e^{i\gamma/2}\begin{pmatrix} \cos(\delta/2) & -e^{-i\epsilon}\sin(\delta/2) \\ e^{i\epsilon}\sin(\delta/2) & \cos(\delta/2) \end{pmatrix}\begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix} \\ &= e^{i\gamma/2}\Big(\cos(\delta/2)\mathbb{1} - i\sin(\delta/2)(\cos(\epsilon)\sigma_y - \sin(\epsilon)\sigma_x)\Big)\mathbf{Z}_\gamma \\ &= e^{i\gamma/2}\mathbf{Z}_\epsilon\mathbf{Y}_\delta\mathbf{Z}_{-\epsilon}\mathbf{Z}_\gamma \\ &= e^{i\gamma/2}\mathbf{Z}_\epsilon\mathbf{Y}_\delta\mathbf{Z}_{\gamma-\epsilon} \end{aligned}$$



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