IQI 04, Seminar 2

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- Qubits' state space.
- Operations on qubits.
- A black box problem.

E. "Manny" Knill: knill@boulder.nist.gov

Qubits' State Space I

- Two qubits, labeled A and B.
  - State space of qubit A:
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- Vector representation of AB's state space.

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- Global phase: $|\psi\rangle_{AB} \simeq e^{i\phi}|\psi\rangle_{AB}$

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  If the states are *distinguishable*. . . . Explanation deferred.

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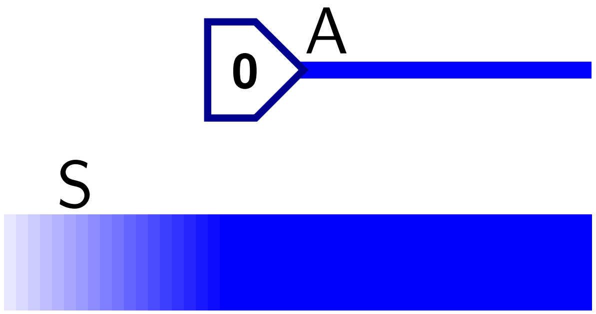
State Preparation

- Prepare a new qubit A in $|0\rangle_{A}$, $\mathbf{prep}(0)^{(A)}$.



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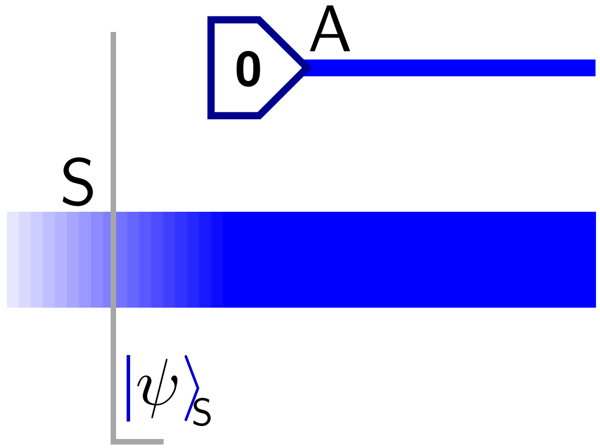

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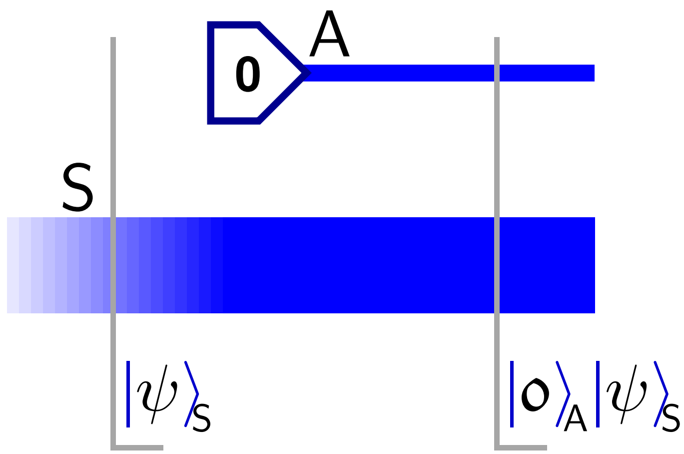

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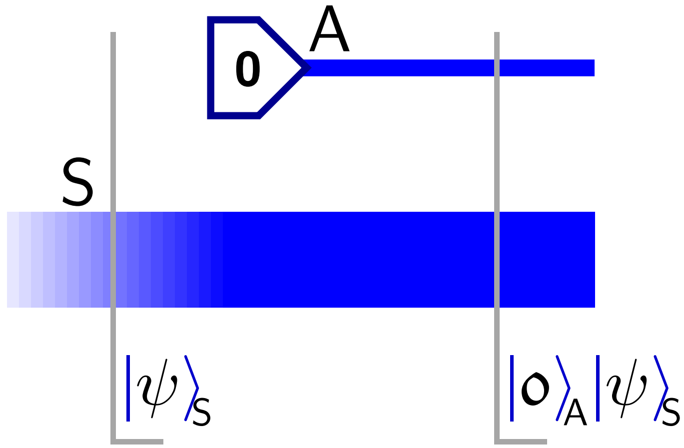

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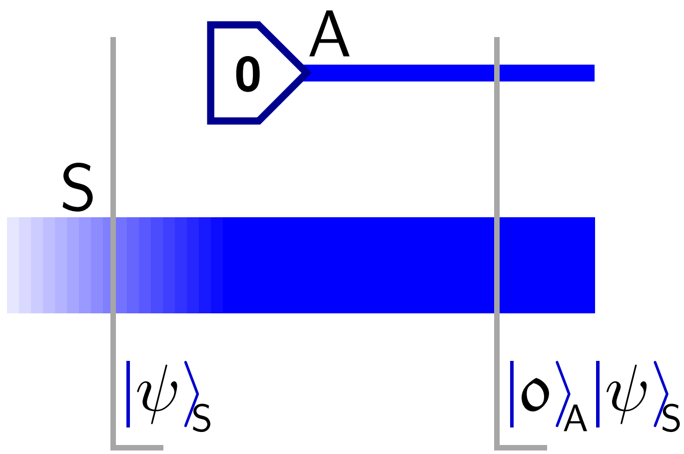

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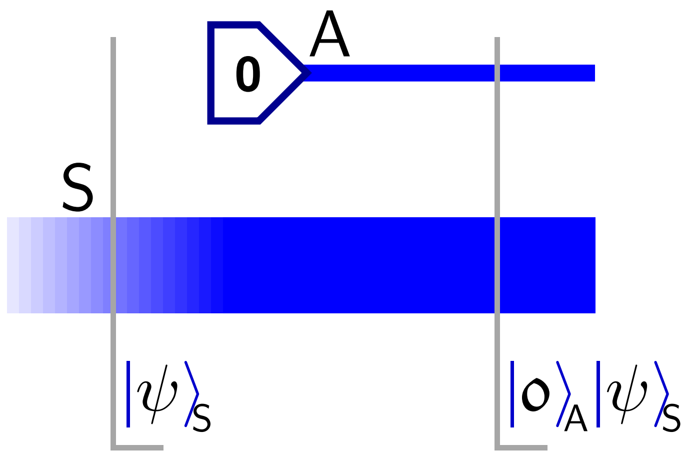

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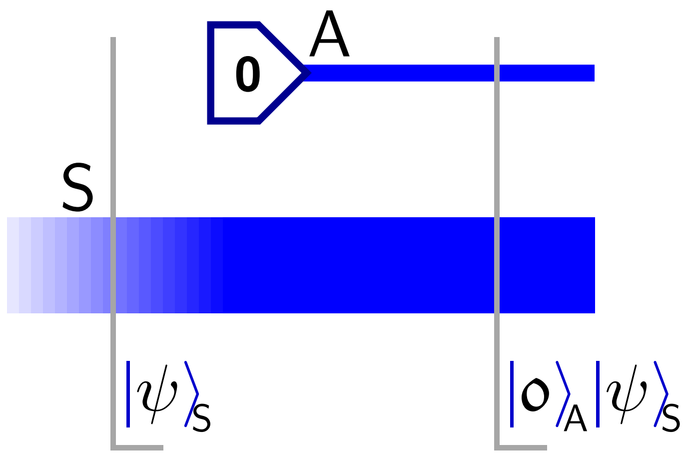

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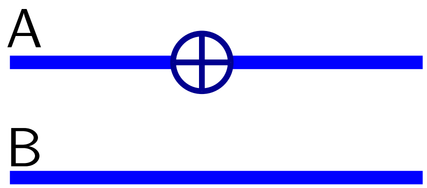
- Apply $\mathbf{not}^{(\mathsf{A})}$ to the two-qubit system AB.



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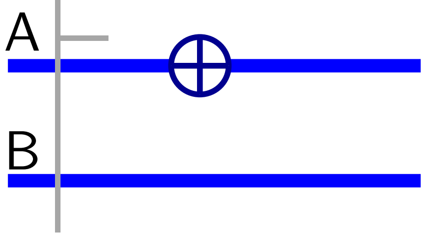

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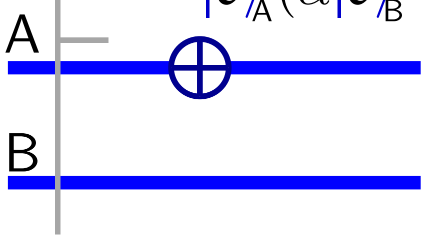

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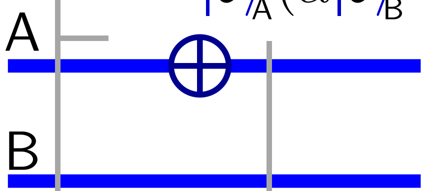

$$\mathbf{not}^{(\mathsf{A})}|\psi\rangle_{\mathsf{AB}} = (\mathbf{not}^{(\mathsf{A})}|0\rangle_{\mathsf{A}})(\alpha|0\rangle_{\mathsf{B}} + \beta|1\rangle_{\mathsf{B}}) + (\mathbf{not}^{(\mathsf{A})}|1\rangle_{\mathsf{A}})(\gamma|0\rangle_{\mathsf{B}} + \delta|1\rangle_{\mathsf{B}})$$

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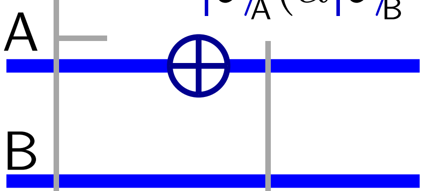

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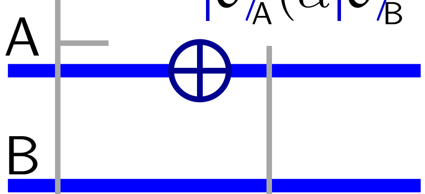

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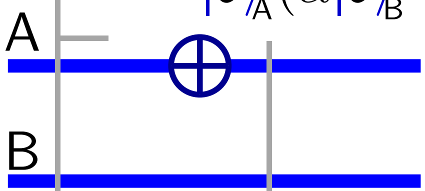
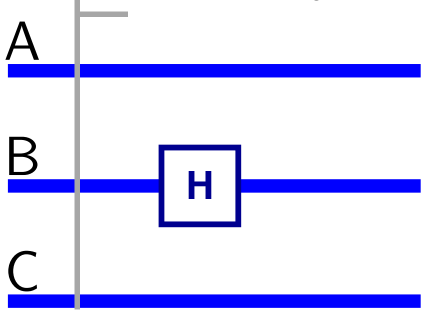
$$
\begin{aligned}
|\psi\rangle_{\mathsf{AB}} &= \alpha|00\rangle_{\mathsf{AB}} + \beta|01\rangle_{\mathsf{AB}} + \gamma|10\rangle_{\mathsf{AB}} + \delta|11\rangle_{\mathsf{AB}} \\
&= |0\rangle_{\mathsf{A}}(\alpha|0\rangle_{\mathsf{B}} + \beta|1\rangle_{\mathsf{B}}) \; + \; |1\rangle_{\mathsf{A}}(\gamma|0\rangle_{\mathsf{B}} + \delta|1\rangle_{\mathsf{B}})
\end{aligned}
$$



$$
\begin{aligned}
\mathbf{not}^{(\mathsf{A})}|\psi\rangle_{\mathsf{AB}} &= (\mathbf{not}^{(\mathsf{A})}|0\rangle_{\mathsf{A}})(\alpha|0\rangle_{\mathsf{B}} + \beta|1\rangle_{\mathsf{B}}) \; + \; (\mathbf{not}^{(\mathsf{A})}|1\rangle_{\mathsf{A}})(\gamma|0\rangle_{\mathsf{B}} + \delta|1\rangle_{\mathsf{B}}) \\
&= |1\rangle_{\mathsf{A}}(\alpha|0\rangle_{\mathsf{B}} + \beta|1\rangle_{\mathsf{B}}) \; + \; |0\rangle_{\mathsf{A}}(\gamma|0\rangle_{\mathsf{B}} + \delta|1\rangle_{\mathsf{B}}) \\
&= \alpha|10\rangle_{\mathsf{AB}} + \beta|11\rangle_{\mathsf{AB}} + \gamma|00\rangle_{\mathsf{AB}} + \delta|01\rangle_{\mathsf{AB}}
\end{aligned}
$$

- Apply $\mathbf{had}^{(\mathsf{B})}$ to the three-qubit system ABC in state

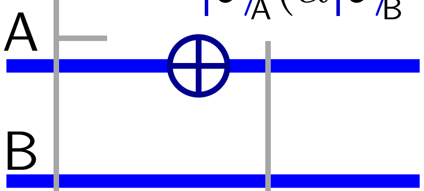
$$
|\psi\rangle_{\mathsf{ABC}} = \tfrac{3}{5}|101\rangle_{\mathsf{ABC}} + \tfrac{4}{5}|110\rangle_{\mathsf{ABC}}.
$$



One-Qubit Gates

- Apply $\mathbf{not}^{(\mathsf{A})}$ to the two-qubit system AB.

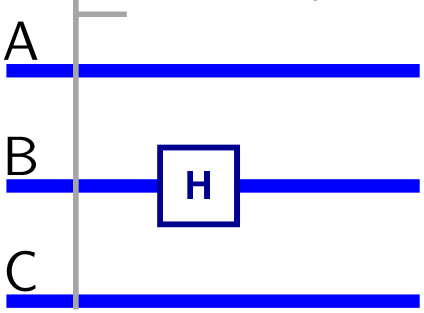
$$|\psi\rangle_{\mathsf{AB}} = \alpha|00\rangle_{\mathsf{AB}} + \beta|01\rangle_{\mathsf{AB}} + \gamma|10\rangle_{\mathsf{AB}} + \delta|11\rangle_{\mathsf{AB}}$$
$$= \ |0\rangle_{\mathsf{A}}(\alpha|0\rangle_{\mathsf{B}} + \beta|1\rangle_{\mathsf{B}}) \ + \ |1\rangle_{\mathsf{A}}(\gamma|0\rangle_{\mathsf{B}} + \delta|1\rangle_{\mathsf{B}})$$



$$\mathbf{not}^{(\mathsf{A})}|\psi\rangle_{\mathsf{AB}} = (\mathbf{not}^{(\mathsf{A})}|0\rangle_{\mathsf{A}})(\alpha|0\rangle_{\mathsf{B}} + \beta|1\rangle_{\mathsf{B}}) \ + \ (\mathbf{not}^{(\mathsf{A})}|1\rangle_{\mathsf{A}})(\gamma|0\rangle_{\mathsf{B}} + \delta|1\rangle_{\mathsf{B}})$$
$$= |1\rangle_{\mathsf{A}}(\alpha|0\rangle_{\mathsf{B}} + \beta|1\rangle_{\mathsf{B}}) \ + \ |0\rangle_{\mathsf{A}}(\gamma|0\rangle_{\mathsf{B}} + \delta|1\rangle_{\mathsf{B}})$$
$$= \alpha|10\rangle_{\mathsf{AB}} + \beta|11\rangle_{\mathsf{AB}} + \gamma|00\rangle_{\mathsf{AB}} + \delta|01\rangle_{\mathsf{AB}}$$

- Apply $\mathbf{had}^{(\mathsf{B})}$ to the three-qubit system ABC in state

$$|\psi\rangle_{\mathsf{ABC}} = \tfrac{3}{5}|101\rangle_{\mathsf{ABC}} + \tfrac{4}{5}|110\rangle_{\mathsf{ABC}}.$$

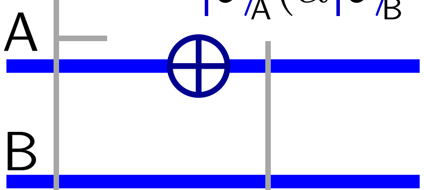


$$\mathbf{had}^{(\mathsf{B})}|\psi\rangle_{\mathsf{ABC}} = \mathbf{had}^{(\mathsf{B})}(\tfrac{3}{5}|101\rangle_{\mathsf{ABC}} + \tfrac{4}{5}|110\rangle_{\mathsf{ABC}})$$

One-Qubit Gates

- Apply $\mathbf{not}^{(A)}$ to the two-qubit system AB.

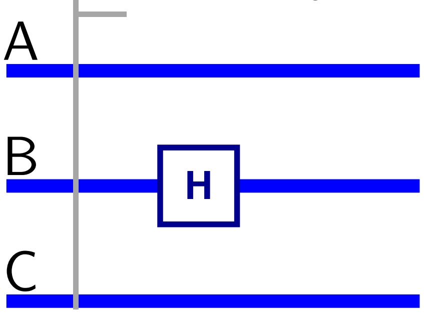
$$|\psi\rangle_{AB} = \alpha|00\rangle_{AB} + \beta|01\rangle_{AB} + \gamma|10\rangle_{AB} + \delta|11\rangle_{AB}$$
$$= |0\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B) + |1\rangle_A(\gamma|0\rangle_B + \delta|1\rangle_B)$$



$$\mathbf{not}^{(A)}|\psi\rangle_{AB} = (\mathbf{not}^{(A)}|0\rangle_A)(\alpha|0\rangle_B + \beta|1\rangle_B) + (\mathbf{not}^{(A)}|1\rangle_A)(\gamma|0\rangle_B + \delta|1\rangle_B)$$
$$= |1\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B) + |0\rangle_A(\gamma|0\rangle_B + \delta|1\rangle_B)$$
$$= \alpha|10\rangle_{AB} + \beta|11\rangle_{AB} + \gamma|00\rangle_{AB} + \delta|01\rangle_{AB}$$

- Apply $\mathbf{had}^{(B)}$ to the three-qubit system ABC in state

$$|\psi\rangle_{ABC} = \tfrac{3}{5}|101\rangle_{ABC} + \tfrac{4}{5}|110\rangle_{ABC}.$$

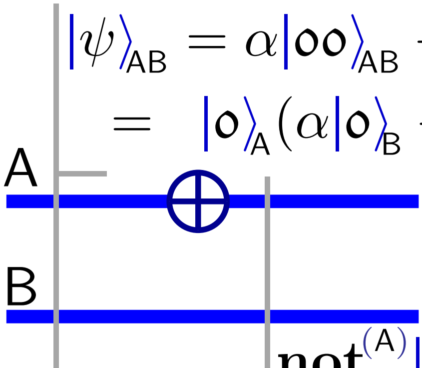


$$\mathbf{had}^{(B)}|\psi\rangle_{ABC} = \mathbf{had}^{(B)}(\tfrac{3}{5}|101\rangle_{ABC} + \tfrac{4}{5}|110\rangle_{ABC})$$
$$= \mathbf{had}^{(B)}(\tfrac{3}{5}|0\rangle_B|1\rangle_A|1\rangle_C + \tfrac{4}{5}|1\rangle_B|1\rangle_A|0\rangle_C)$$

One-Qubit Gates

- Apply $\mathbf{not}^{(A)}$ to the two-qubit system AB.

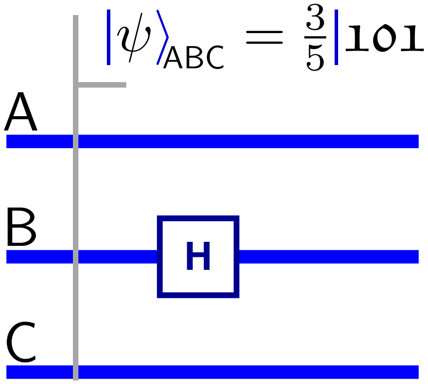
$$|\psi\rangle_{AB} = \alpha|00\rangle_{AB} + \beta|01\rangle_{AB} + \gamma|10\rangle_{AB} + \delta|11\rangle_{AB}$$
$$= |0\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B) + |1\rangle_A(\gamma|0\rangle_B + \delta|1\rangle_B)$$



$$\mathbf{not}^{(A)}|\psi\rangle_{AB} = (\mathbf{not}^{(A)}|0\rangle_A)(\alpha|0\rangle_B + \beta|1\rangle_B) + (\mathbf{not}^{(A)}|1\rangle_A)(\gamma|0\rangle_B + \delta|1\rangle_B)$$
$$= |1\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B) + |0\rangle_A(\gamma|0\rangle_B + \delta|1\rangle_B)$$
$$= \alpha|10\rangle_{AB} + \beta|11\rangle_{AB} + \gamma|00\rangle_{AB} + \delta|01\rangle_{AB}$$

- Apply $\mathbf{had}^{(B)}$ to the three-qubit system ABC in state

$$|\psi\rangle_{ABC} = \tfrac{3}{5}|101\rangle_{ABC} + \tfrac{4}{5}|110\rangle_{ABC}.$$

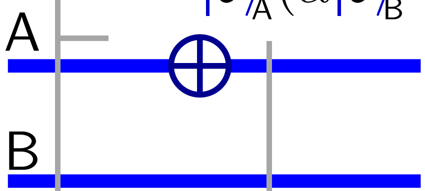


$$\mathbf{had}^{(B)}|\psi\rangle_{ABC} = \mathbf{had}^{(B)}(\tfrac{3}{5}|101\rangle_{ABC} + \tfrac{4}{5}|110\rangle_{ABC})$$
$$= \mathbf{had}^{(B)}(\tfrac{3}{5}|0\rangle_B|1\rangle_A|1\rangle_C + \tfrac{4}{5}|1\rangle_B|1\rangle_A|0\rangle_C)$$
$$= \tfrac{3}{5}\mathbf{had}^{(B)}|0\rangle_B|1\rangle_A|1\rangle_C + \tfrac{4}{5}\mathbf{had}^{(B)}|1\rangle_B|1\rangle_A|0\rangle_C$$

One-Qubit Gates

- Apply $\mathbf{not}^{(A)}$ to the two-qubit system AB.

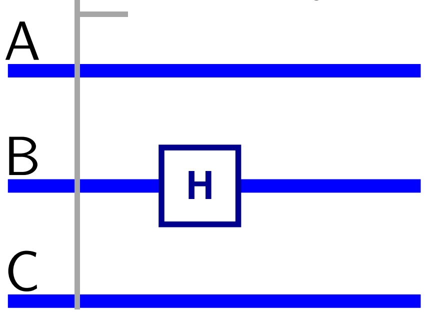
$$|\psi\rangle_{AB} = \alpha|00\rangle_{AB} + \beta|01\rangle_{AB} + \gamma|10\rangle_{AB} + \delta|11\rangle_{AB}$$
$$= |0\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B) + |1\rangle_A(\gamma|0\rangle_B + \delta|1\rangle_B)$$



$$\mathbf{not}^{(A)}|\psi\rangle_{AB} = (\mathbf{not}^{(A)}|0\rangle_A)(\alpha|0\rangle_B + \beta|1\rangle_B) + (\mathbf{not}^{(A)}|1\rangle_A)(\gamma|0\rangle_B + \delta|1\rangle_B)$$
$$= |1\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B) + |0\rangle_A(\gamma|0\rangle_B + \delta|1\rangle_B)$$
$$= \alpha|10\rangle_{AB} + \beta|11\rangle_{AB} + \gamma|00\rangle_{AB} + \delta|01\rangle_{AB}$$

- Apply $\mathbf{had}^{(B)}$ to the three-qubit system ABC in state

$$|\psi\rangle_{ABC} = \tfrac{3}{5}|101\rangle_{ABC} + \tfrac{4}{5}|110\rangle_{ABC}.$$



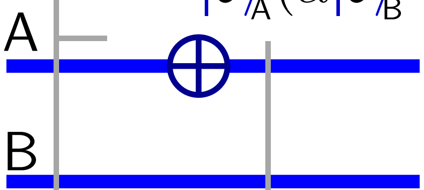
$$\mathbf{had}^{(B)}|\psi\rangle_{ABC} = \mathbf{had}^{(B)}(\tfrac{3}{5}|101\rangle_{ABC} + \tfrac{4}{5}|110\rangle_{ABC})$$

$$= \tfrac{3}{5}\mathbf{had}^{(B)}|0\rangle_B|1\rangle_A|1\rangle_C + \tfrac{4}{5}\mathbf{had}^{(B)}|1\rangle_B|1\rangle_A|0\rangle_C$$

One-Qubit Gates

- Apply $\mathbf{not}^{(\mathsf{A})}$ to the two-qubit system AB.

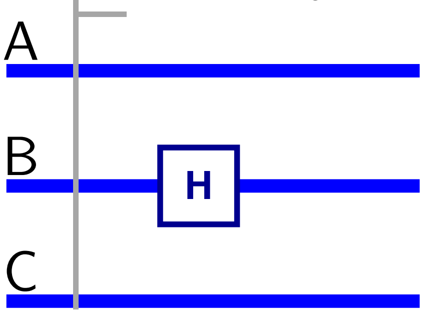
$$
\begin{aligned}
|\psi\rangle_{\mathsf{AB}} &= \alpha|00\rangle_{\mathsf{AB}} + \beta|01\rangle_{\mathsf{AB}} + \gamma|10\rangle_{\mathsf{AB}} + \delta|11\rangle_{\mathsf{AB}} \\
&= |0\rangle_{\mathsf{A}}(\alpha|0\rangle_{\mathsf{B}} + \beta|1\rangle_{\mathsf{B}}) \; + \; |1\rangle_{\mathsf{A}}(\gamma|0\rangle_{\mathsf{B}} + \delta|1\rangle_{\mathsf{B}})
\end{aligned}
$$



$$
\begin{aligned}
\mathbf{not}^{(\mathsf{A})}|\psi\rangle_{\mathsf{AB}} &= (\mathbf{not}^{(\mathsf{A})}|0\rangle_{\mathsf{A}})(\alpha|0\rangle_{\mathsf{B}} + \beta|1\rangle_{\mathsf{B}}) \; + \; (\mathbf{not}^{(\mathsf{A})}|1\rangle_{\mathsf{A}})(\gamma|0\rangle_{\mathsf{B}} + \delta|1\rangle_{\mathsf{B}}) \\
&= |1\rangle_{\mathsf{A}}(\alpha|0\rangle_{\mathsf{B}} + \beta|1\rangle_{\mathsf{B}}) \; + \; |0\rangle_{\mathsf{A}}(\gamma|0\rangle_{\mathsf{B}} + \delta|1\rangle_{\mathsf{B}}) \\
&= \alpha|10\rangle_{\mathsf{AB}} + \beta|11\rangle_{\mathsf{AB}} + \gamma|00\rangle_{\mathsf{AB}} + \delta|01\rangle_{\mathsf{AB}}
\end{aligned}
$$

- Apply $\mathbf{had}^{(\mathsf{B})}$ to the three-qubit system ABC in state

$$|\psi\rangle_{\mathsf{ABC}} = \tfrac{3}{5}|101\rangle_{\mathsf{ABC}} + \tfrac{4}{5}|110\rangle_{\mathsf{ABC}}.$$

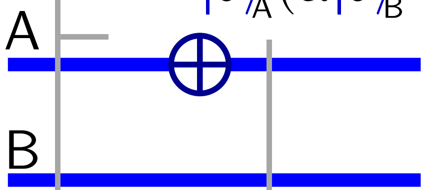


$$
\begin{aligned}
\mathbf{had}^{(\mathsf{B})}|\psi\rangle_{\mathsf{ABC}} &= \mathbf{had}^{(\mathsf{B})}(\tfrac{3}{5}|101\rangle_{\mathsf{ABC}} + \tfrac{4}{5}|110\rangle_{\mathsf{ABC}}) \\
&= \tfrac{3}{5}\mathbf{had}^{(\mathsf{B})}|0\rangle_{\mathsf{B}}|1\rangle_{\mathsf{A}}|1\rangle_{\mathsf{C}} + \tfrac{4}{5}\mathbf{had}^{(\mathsf{B})}|1\rangle_{\mathsf{B}}|1\rangle_{\mathsf{A}}|0\rangle_{\mathsf{C}}
\end{aligned}
$$

One-Qubit Gates

- Apply $\mathbf{not}^{(\mathsf{A})}$ to the two-qubit system AB.

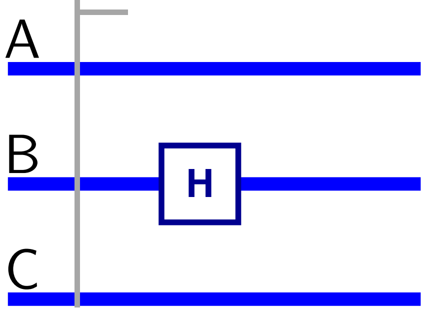
$$|\psi\rangle_{\mathsf{AB}} = \alpha|00\rangle_{\mathsf{AB}} + \beta|01\rangle_{\mathsf{AB}} + \gamma|10\rangle_{\mathsf{AB}} + \delta|11\rangle_{\mathsf{AB}}$$
$$= \; |0\rangle_{\mathsf{A}}(\alpha|0\rangle_{\mathsf{B}} + \beta|1\rangle_{\mathsf{B}}) \; + \; |1\rangle_{\mathsf{A}}(\gamma|0\rangle_{\mathsf{B}} + \delta|1\rangle_{\mathsf{B}})$$



$$\mathbf{not}^{(\mathsf{A})}|\psi\rangle_{\mathsf{AB}} = (\mathbf{not}^{(\mathsf{A})}|0\rangle_{\mathsf{A}})(\alpha|0\rangle_{\mathsf{B}} + \beta|1\rangle_{\mathsf{B}}) \; + \; (\mathbf{not}^{(\mathsf{A})}|1\rangle_{\mathsf{A}})(\gamma|0\rangle_{\mathsf{B}} + \delta|1\rangle_{\mathsf{B}})$$
$$= |1\rangle_{\mathsf{A}}(\alpha|0\rangle_{\mathsf{B}} + \beta|1\rangle_{\mathsf{B}}) \; + \; |0\rangle_{\mathsf{A}}(\gamma|0\rangle_{\mathsf{B}} + \delta|1\rangle_{\mathsf{B}})$$
$$= \alpha|10\rangle_{\mathsf{AB}} + \beta|11\rangle_{\mathsf{AB}} + \gamma|00\rangle_{\mathsf{AB}} + \delta|01\rangle_{\mathsf{AB}}$$

- Apply $\mathbf{had}^{(\mathsf{B})}$ to the three-qubit system ABC in state

$$|\psi\rangle_{\mathsf{ABC}} = \tfrac{3}{5}|101\rangle_{\mathsf{ABC}} + \tfrac{4}{5}|110\rangle_{\mathsf{ABC}}.$$

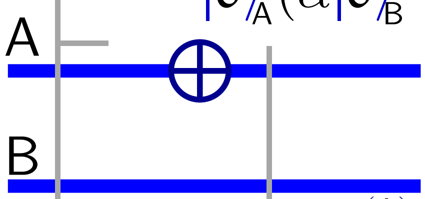


$$\mathbf{had}^{(\mathsf{B})}|\psi\rangle_{\mathsf{ABC}} = \mathbf{had}^{(\mathsf{B})}(\tfrac{3}{5}|101\rangle_{\mathsf{ABC}} + \tfrac{4}{5}|110\rangle_{\mathsf{ABC}})$$
$$= \tfrac{3}{5}\mathbf{had}^{(\mathsf{B})}|0\rangle_{\mathsf{B}}|1\rangle_{\mathsf{A}}|1\rangle_{\mathsf{C}} + \tfrac{4}{5}\mathbf{had}^{(\mathsf{B})}|1\rangle_{\mathsf{B}}|1\rangle_{\mathsf{A}}|0\rangle_{\mathsf{C}}$$
$$= \frac{3}{5}\frac{1}{\sqrt{2}}(|0\rangle_{\mathsf{B}} + |1\rangle_{\mathsf{B}})|1\rangle_{\mathsf{A}}|1\rangle_{\mathsf{C}} + \frac{4}{5}\frac{1}{\sqrt{2}}(|0\rangle_{\mathsf{B}} - |1\rangle_{\mathsf{B}})|1\rangle_{\mathsf{A}}|0\rangle_{\mathsf{C}}$$

One-Qubit Gates

- Apply $\mathbf{not}^{(A)}$ to the two-qubit system AB.

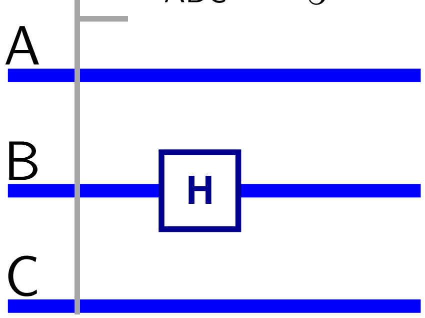
$$|\psi\rangle_{AB} = \alpha|00\rangle_{AB} + \beta|01\rangle_{AB} + \gamma|10\rangle_{AB} + \delta|11\rangle_{AB}$$
$$= |0\rangle_{A}(\alpha|0\rangle_{B} + \beta|1\rangle_{B}) \;+\; |1\rangle_{A}(\gamma|0\rangle_{B} + \delta|1\rangle_{B})$$



$$\mathbf{not}^{(A)}|\psi\rangle_{AB} = (\mathbf{not}^{(A)}|0\rangle_{A})(\alpha|0\rangle_{B} + \beta|1\rangle_{B}) \;+\; (\mathbf{not}^{(A)}|1\rangle_{A})(\gamma|0\rangle_{B} + \delta|1\rangle_{B})$$
$$= |1\rangle_{A}(\alpha|0\rangle_{B} + \beta|1\rangle_{B}) \;+\; |0\rangle_{A}(\gamma|0\rangle_{B} + \delta|1\rangle_{B})$$
$$= \alpha|10\rangle_{AB} + \beta|11\rangle_{AB} + \gamma|00\rangle_{AB} + \delta|01\rangle_{AB}$$

- Apply $\mathbf{had}^{(B)}$ to the three-qubit system ABC in state

$$|\psi\rangle_{ABC} = \tfrac{3}{5}|101\rangle_{ABC} + \tfrac{4}{5}|110\rangle_{ABC}.$$

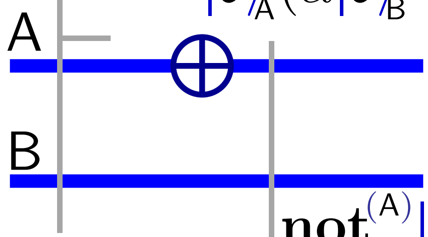


$$\mathbf{had}^{(B)}|\psi\rangle_{ABC} = \mathbf{had}^{(B)}(\tfrac{3}{5}|101\rangle_{ABC} + \tfrac{4}{5}|110\rangle_{ABC})$$
$$= \tfrac{3}{5}\tfrac{1}{\sqrt{2}}(|0\rangle_{B} + |1\rangle_{B})|1\rangle_{A}|1\rangle_{C} + \tfrac{4}{5}\tfrac{1}{\sqrt{2}}(|0\rangle_{B} - |1\rangle_{B})|1\rangle_{A}|0\rangle_{C}$$

One-Qubit Gates

- Apply $\mathbf{not}^{(\mathsf{A})}$ to the two-qubit system AB.

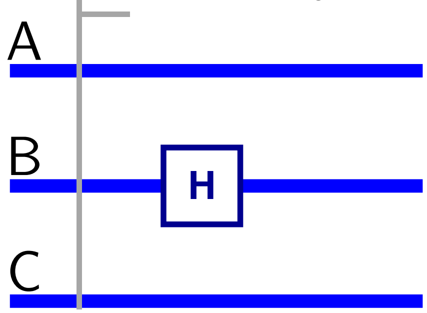
$$|\psi\rangle_{\mathsf{AB}} = \alpha|00\rangle_{\mathsf{AB}} + \beta|01\rangle_{\mathsf{AB}} + \gamma|10\rangle_{\mathsf{AB}} + \delta|11\rangle_{\mathsf{AB}}$$
$$= |0\rangle_{\mathsf{A}}(\alpha|0\rangle_{\mathsf{B}} + \beta|1\rangle_{\mathsf{B}}) + |1\rangle_{\mathsf{A}}(\gamma|0\rangle_{\mathsf{B}} + \delta|1\rangle_{\mathsf{B}})$$



$$\mathbf{not}^{(\mathsf{A})}|\psi\rangle_{\mathsf{AB}} = (\mathbf{not}^{(\mathsf{A})}|0\rangle_{\mathsf{A}})(\alpha|0\rangle_{\mathsf{B}} + \beta|1\rangle_{\mathsf{B}}) + (\mathbf{not}^{(\mathsf{A})}|1\rangle_{\mathsf{A}})(\gamma|0\rangle_{\mathsf{B}} + \delta|1\rangle_{\mathsf{B}})$$
$$= |1\rangle_{\mathsf{A}}(\alpha|0\rangle_{\mathsf{B}} + \beta|1\rangle_{\mathsf{B}}) + |0\rangle_{\mathsf{A}}(\gamma|0\rangle_{\mathsf{B}} + \delta|1\rangle_{\mathsf{B}})$$
$$= \alpha|10\rangle_{\mathsf{AB}} + \beta|11\rangle_{\mathsf{AB}} + \gamma|00\rangle_{\mathsf{AB}} + \delta|01\rangle_{\mathsf{AB}}$$

- Apply $\mathbf{had}^{(\mathsf{B})}$ to the three-qubit system ABC in state

$$|\psi\rangle_{\mathsf{ABC}} = \tfrac{3}{5}|101\rangle_{\mathsf{ABC}} + \tfrac{4}{5}|110\rangle_{\mathsf{ABC}}.$$

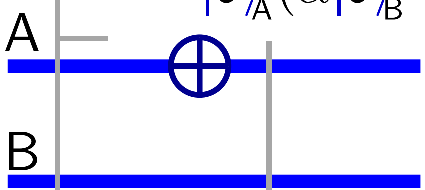


$$\mathbf{had}^{(\mathsf{B})}|\psi\rangle_{\mathsf{ABC}} = \mathbf{had}^{(\mathsf{B})}(\tfrac{3}{5}|101\rangle_{\mathsf{ABC}} + \tfrac{4}{5}|110\rangle_{\mathsf{ABC}})$$
$$= \frac{3}{5}\frac{1}{\sqrt{2}}(|0\rangle_{\mathsf{B}} + |1\rangle_{\mathsf{B}})|1\rangle_{\mathsf{A}}|1\rangle_{\mathsf{C}} + \frac{4}{5}\frac{1}{\sqrt{2}}(|0\rangle_{\mathsf{B}} - |1\rangle_{\mathsf{B}})|1\rangle_{\mathsf{A}}|0\rangle_{\mathsf{C}}$$

One-Qubit Gates

- Apply $\mathbf{not}^{(A)}$ to the two-qubit system AB.

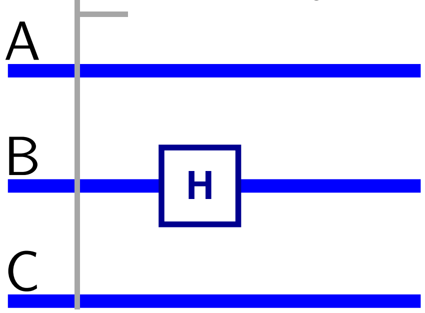
$$|\psi\rangle_{AB} = \alpha|00\rangle_{AB} + \beta|01\rangle_{AB} + \gamma|10\rangle_{AB} + \delta|11\rangle_{AB}$$
$$= |0\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B) + |1\rangle_A(\gamma|0\rangle_B + \delta|1\rangle_B)$$



$$\mathbf{not}^{(A)}|\psi\rangle_{AB} = (\mathbf{not}^{(A)}|0\rangle_A)(\alpha|0\rangle_B + \beta|1\rangle_B) + (\mathbf{not}^{(A)}|1\rangle_A)(\gamma|0\rangle_B + \delta|1\rangle_B)$$
$$= |1\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B) + |0\rangle_A(\gamma|0\rangle_B + \delta|1\rangle_B)$$
$$= \alpha|10\rangle_{AB} + \beta|11\rangle_{AB} + \gamma|00\rangle_{AB} + \delta|01\rangle_{AB}$$

- Apply $\mathbf{had}^{(B)}$ to the three-qubit system ABC in state

$$|\psi\rangle_{ABC} = \tfrac{3}{5}|101\rangle_{ABC} + \tfrac{4}{5}|110\rangle_{ABC}.$$

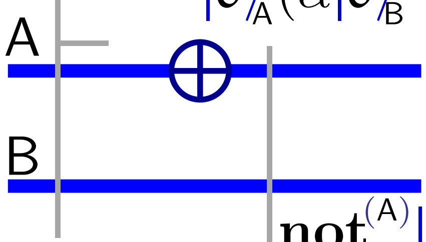


$$\mathbf{had}^{(B)}|\psi\rangle_{ABC} = \mathbf{had}^{(B)}(\tfrac{3}{5}|101\rangle_{ABC} + \tfrac{4}{5}|110\rangle_{ABC})$$
$$= \frac{3}{5}\frac{1}{\sqrt{2}}(|0\rangle_B + |1\rangle_B)|1\rangle_A|1\rangle_C + \frac{4}{5}\frac{1}{\sqrt{2}}(|0\rangle_B - |1\rangle_B)|1\rangle_A|0\rangle_C$$
$$= \frac{3}{5\sqrt{2}}(|1\rangle_A|0\rangle_B|1\rangle_C + |1\rangle_A|1\rangle_B|1\rangle_C)$$
$$+ \frac{4}{5\sqrt{2}}(|1\rangle_A|0\rangle_B|0\rangle_C - |1\rangle_A|1\rangle_B|0\rangle_C)$$

One-Qubit Gates

- Apply $\mathbf{not}^{(\mathsf{A})}$ to the two-qubit system AB.

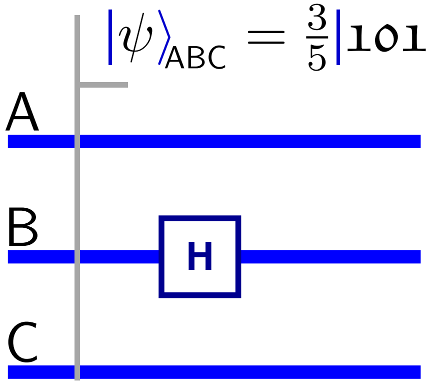
$$|\psi\rangle_{AB} = \alpha|00\rangle_{AB} + \beta|01\rangle_{AB} + \gamma|10\rangle_{AB} + \delta|11\rangle_{AB}$$
$$= |0\rangle_{A}(\alpha|0\rangle_{B} + \beta|1\rangle_{B}) + |1\rangle_{A}(\gamma|0\rangle_{B} + \delta|1\rangle_{B})$$



$$\mathbf{not}^{(A)}|\psi\rangle_{AB} = (\mathbf{not}^{(A)}|0\rangle_{A})(\alpha|0\rangle_{B} + \beta|1\rangle_{B}) + (\mathbf{not}^{(A)}|1\rangle_{A})(\gamma|0\rangle_{B} + \delta|1\rangle_{B})$$
$$= |1\rangle_{A}(\alpha|0\rangle_{B} + \beta|1\rangle_{B}) + |0\rangle_{A}(\gamma|0\rangle_{B} + \delta|1\rangle_{B})$$
$$= \alpha|10\rangle_{AB} + \beta|11\rangle_{AB} + \gamma|00\rangle_{AB} + \delta|01\rangle_{AB}$$

- Apply $\mathbf{had}^{(\mathsf{B})}$ to the three-qubit system ABC in state

$$|\psi\rangle_{ABC} = \tfrac{3}{5}|101\rangle_{ABC} + \tfrac{4}{5}|110\rangle_{ABC}.$$

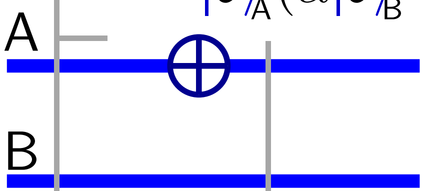


$$\mathbf{had}^{(B)}|\psi\rangle_{ABC} = \mathbf{had}^{(B)}(\tfrac{3}{5}|101\rangle_{ABC} + \tfrac{4}{5}|110\rangle_{ABC})$$
$$= \frac{3}{5\sqrt{2}}(|1\rangle_{A}|0\rangle_{B}|1\rangle_{C} + |1\rangle_{A}|1\rangle_{B}|1\rangle_{C})$$
$$+ \frac{4}{5\sqrt{2}}(|1\rangle_{A}|0\rangle_{B}|0\rangle_{C} - |1\rangle_{A}|1\rangle_{B}|0\rangle_{C})$$

One-Qubit Gates

- Apply $\mathbf{not}^{(\mathsf{A})}$ to the two-qubit system AB.

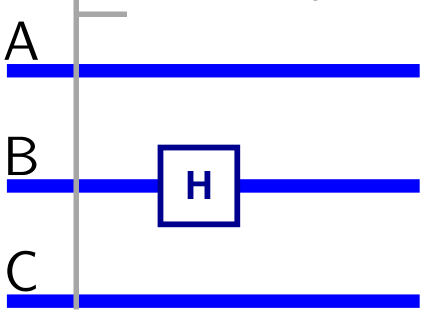
$$
\begin{aligned}
|\psi\rangle_{AB} &= \alpha|00\rangle_{AB} + \beta|01\rangle_{AB} + \gamma|10\rangle_{AB} + \delta|11\rangle_{AB} \\
&= |0\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B) \; + \; |1\rangle_A(\gamma|0\rangle_B + \delta|1\rangle_B)
\end{aligned}
$$



$$
\begin{aligned}
\mathbf{not}^{(A)}|\psi\rangle_{AB} &= (\mathbf{not}^{(A)}|0\rangle_A)(\alpha|0\rangle_B + \beta|1\rangle_B) \; + \; (\mathbf{not}^{(A)}|1\rangle_A)(\gamma|0\rangle_B + \delta|1\rangle_B) \\
&= |1\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B) \; + \; |0\rangle_A(\gamma|0\rangle_B + \delta|1\rangle_B) \\
&= \alpha|10\rangle_{AB} + \beta|11\rangle_{AB} + \gamma|00\rangle_{AB} + \delta|01\rangle_{AB}
\end{aligned}
$$

- Apply $\mathbf{had}^{(\mathsf{B})}$ to the three-qubit system ABC in state

$$
|\psi\rangle_{ABC} = \tfrac{3}{5}|101\rangle_{ABC} + \tfrac{4}{5}|110\rangle_{ABC}.
$$

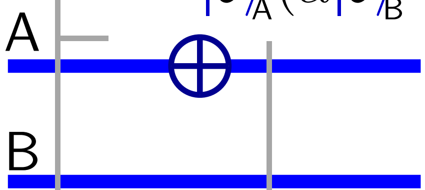


$$
\begin{aligned}
\mathbf{had}^{(B)}|\psi\rangle_{ABC} &= \mathbf{had}^{(B)}(\tfrac{3}{5}|101\rangle_{ABC} + \tfrac{4}{5}|110\rangle_{ABC}) \\
&= \frac{3}{5\sqrt{2}}(|1\rangle_A|0\rangle_B|1\rangle_C + |1\rangle_A|1\rangle_B|1\rangle_C) \\
&\quad + \frac{4}{5\sqrt{2}}(|1\rangle_A|0\rangle_B|0\rangle_C - |1\rangle_A|1\rangle_B|0\rangle_C)
\end{aligned}
$$

One-Qubit Gates

- Apply $\mathbf{not}^{(A)}$ to the two-qubit system AB.

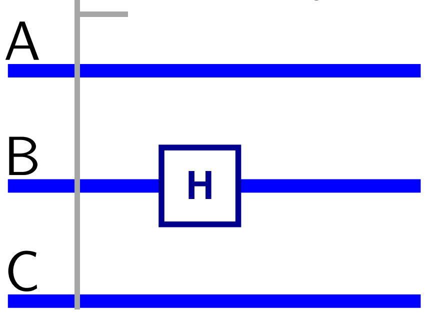
$$|\psi\rangle_{AB} = \alpha|00\rangle_{AB} + \beta|01\rangle_{AB} + \gamma|10\rangle_{AB} + \delta|11\rangle_{AB}$$
$$= \ |0\rangle_{A}(\alpha|0\rangle_{B} + \beta|1\rangle_{B}) \ + \ |1\rangle_{A}(\gamma|0\rangle_{B} + \delta|1\rangle_{B})$$



$$\begin{aligned}\mathbf{not}^{(A)}|\psi\rangle_{AB} &= (\mathbf{not}^{(A)}|0\rangle_{A})(\alpha|0\rangle_{B} + \beta|1\rangle_{B}) \ + \ (\mathbf{not}^{(A)}|1\rangle_{A})(\gamma|0\rangle_{B} + \delta|1\rangle_{B}) \\ &= |1\rangle_{A}(\alpha|0\rangle_{B} + \beta|1\rangle_{B}) \ + \ |0\rangle_{A}(\gamma|0\rangle_{B} + \delta|1\rangle_{B}) \\ &= \alpha|10\rangle_{AB} + \beta|11\rangle_{AB} + \gamma|00\rangle_{AB} + \delta|01\rangle_{AB}\end{aligned}$$

- Apply $\mathbf{had}^{(B)}$ to the three-qubit system ABC in state

$$|\psi\rangle_{ABC} = \tfrac{3}{5}|101\rangle_{ABC} + \tfrac{4}{5}|110\rangle_{ABC}.$$



$$\begin{aligned}\mathbf{had}^{(B)}|\psi\rangle_{ABC} &= \mathbf{had}^{(B)}(\tfrac{3}{5}|101\rangle_{ABC} + \tfrac{4}{5}|110\rangle_{ABC}) \\ &= \frac{3}{5\sqrt{2}}(|1\rangle_{A}|0\rangle_{B}|1\rangle_{C} + |1\rangle_{A}|1\rangle_{B}|1\rangle_{C}) \\ &\quad + \frac{4}{5\sqrt{2}}(|1\rangle_{A}|0\rangle_{B}|0\rangle_{C} - |1\rangle_{A}|1\rangle_{B}|0\rangle_{C}) \\ &= \frac{3}{5\sqrt{2}}(|101\rangle_{ABC} + |111\rangle_{ABC}) + \frac{4}{5\sqrt{2}}(|100\rangle_{ABC} - |110\rangle_{ABC})\end{aligned}$$

Controlled-Not

- The controlled not acts on two qubits: **if** A **then** $\mathbf{not}^{(B)}$.

Controlled-Not

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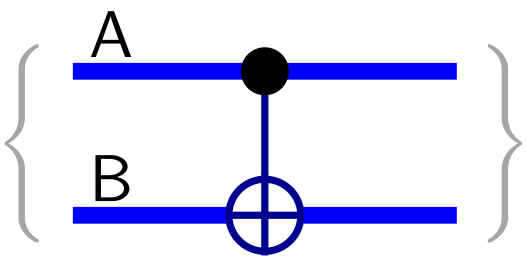

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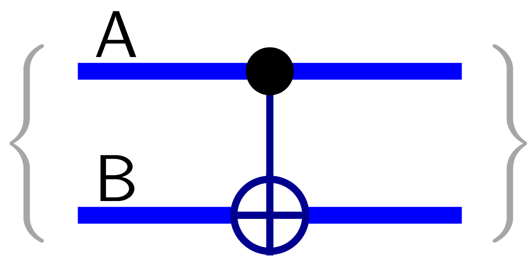
$|00\rangle_{AB} \longrightarrow$



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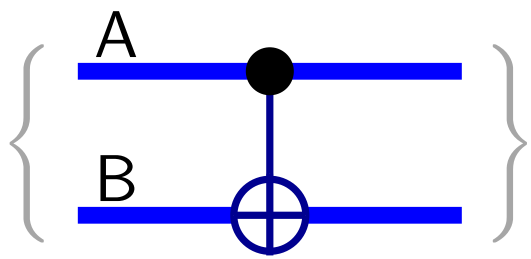

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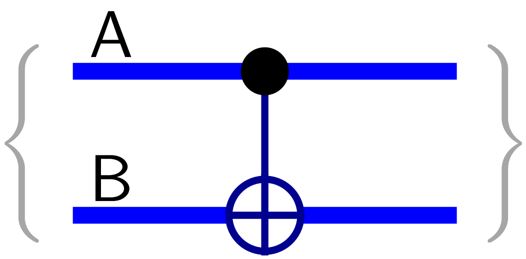

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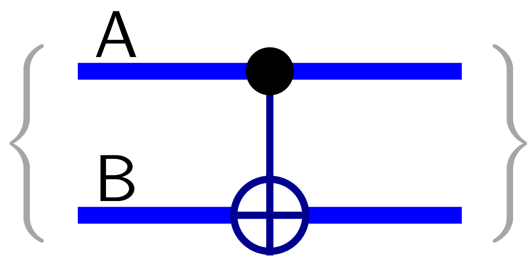

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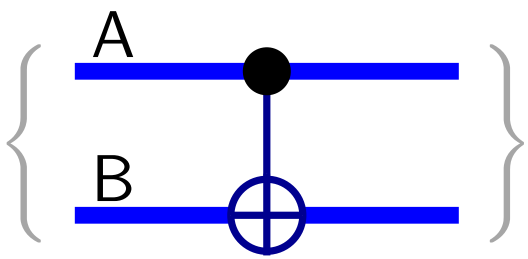

$$\mathbf{cnot}^{(\mathsf{AB})}|\mathfrak{a}\mathfrak{b}\rangle_{\mathsf{AB}} = |\mathfrak{a}(\mathfrak{b}+\mathfrak{a})\rangle_{\mathsf{AB}}$$

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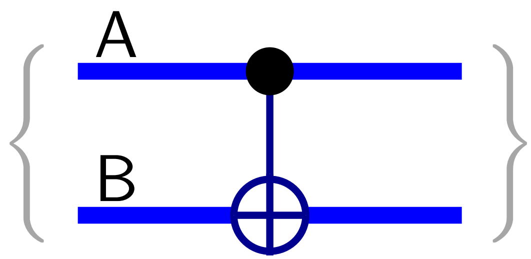

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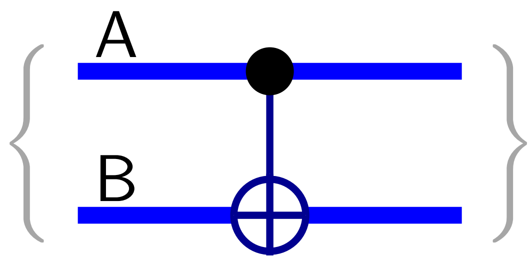

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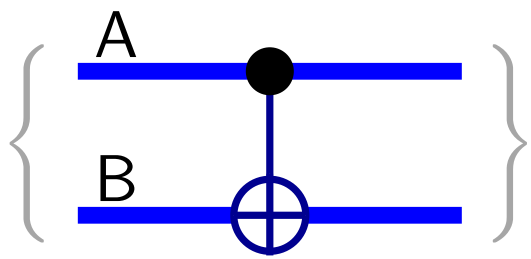

$$\begin{array}{l} |00\rangle_{AB} \\ |01\rangle_{AB} \\ |11\rangle_{AB} \\ |10\rangle_{AB} \end{array}$$

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- Example:

$$\begin{aligned} \mathbf{cnot}^{(AB)}(\ldots + \beta|010\rangle_{SAB} + \gamma|100\rangle_{SAB} + \ldots) \\ = (\ldots + \beta \end{aligned}$$

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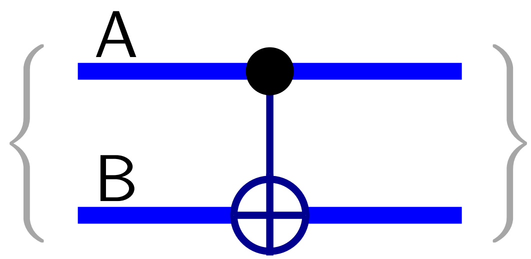

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$$\mathbf{cnot}^{(\mathsf{AB})}(\ldots + \beta|010\rangle_{\mathsf{SAB}} + \gamma|100\rangle_{\mathsf{SAB}} + \ldots)$$
$$= \ (\ldots + \beta|011\rangle_{\mathsf{SAB}} + \gamma$$

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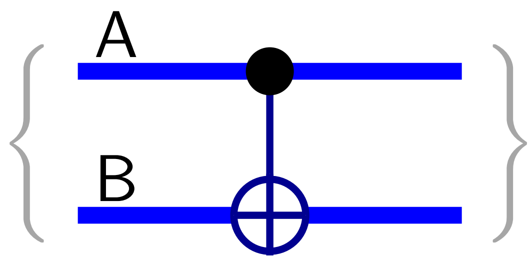
$$\begin{array}{ll} |00\rangle_{AB} \longrightarrow & |00\rangle_{AB} \\ |01\rangle_{AB} \longrightarrow & |01\rangle_{AB} \\ |10\rangle_{AB} \longrightarrow & |11\rangle_{AB} \\ |11\rangle_{AB} \longrightarrow & |10\rangle_{AB} \end{array}$$



$$\mathbf{cnot}^{(AB)}|\mathfrak{a}\mathfrak{b}\rangle_{AB} = |\mathfrak{a}(\mathfrak{b}+\mathfrak{a})\rangle_{AB}$$ ... "+" is modulo 2: $\left\{\begin{array}{lll} 0+0 & = & 0 \\ 0+1 & = & 1 \\ 1+0 & = & 1 \\ 1+1 & = & 0 \end{array}\right.$

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$$
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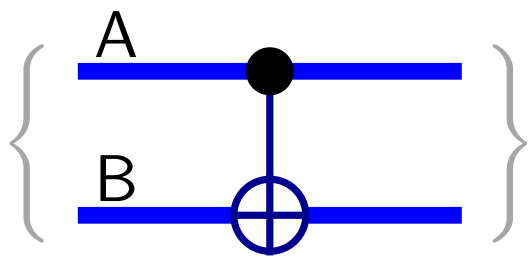
\end{array}
$$



$$
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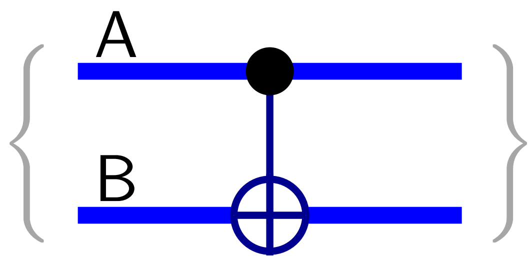

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- The controlled not acts on two qubits: **if** A **then** $\mathbf{not}^{(B)}$.

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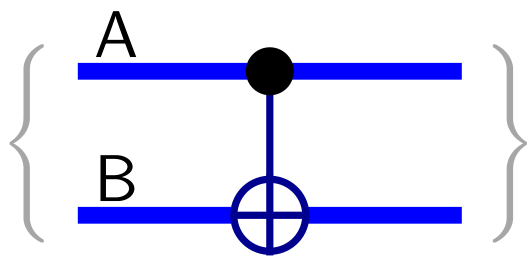
|10\rangle_{AB} &\rightarrow \\
|11\rangle_{AB} &\rightarrow
\end{aligned}$$



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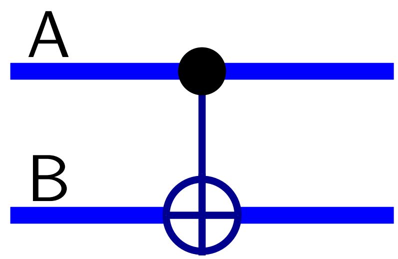

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- Problem: Prepare the state $\frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$.

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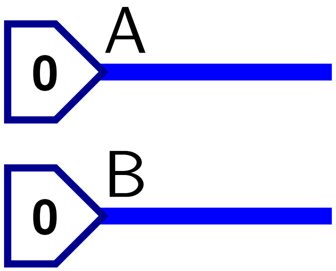

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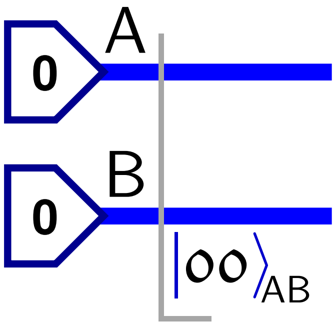

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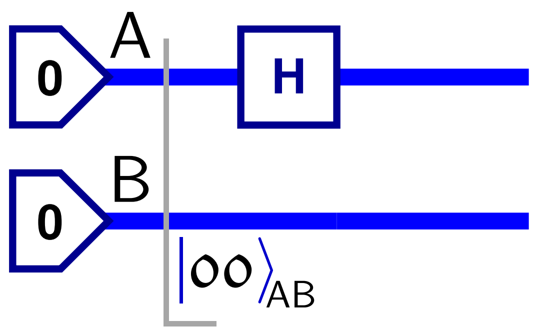

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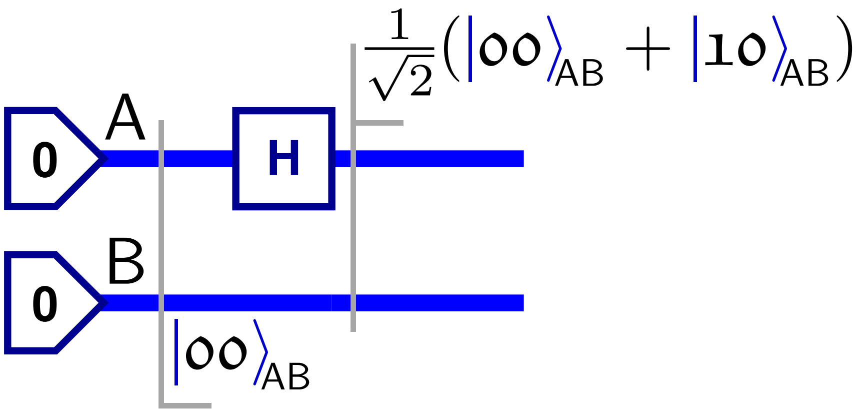

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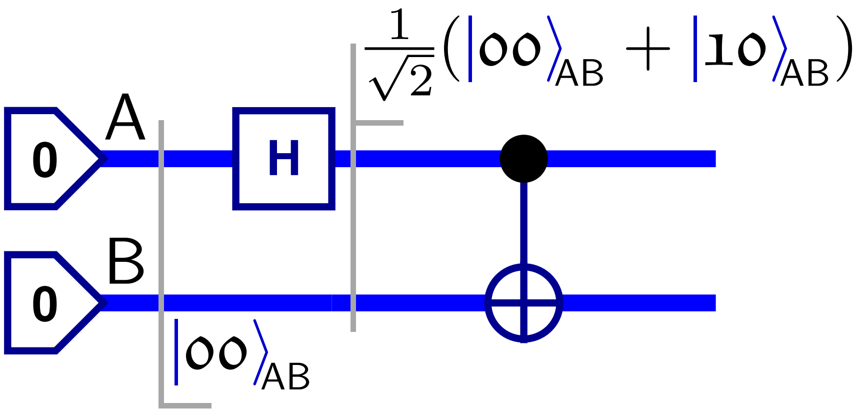

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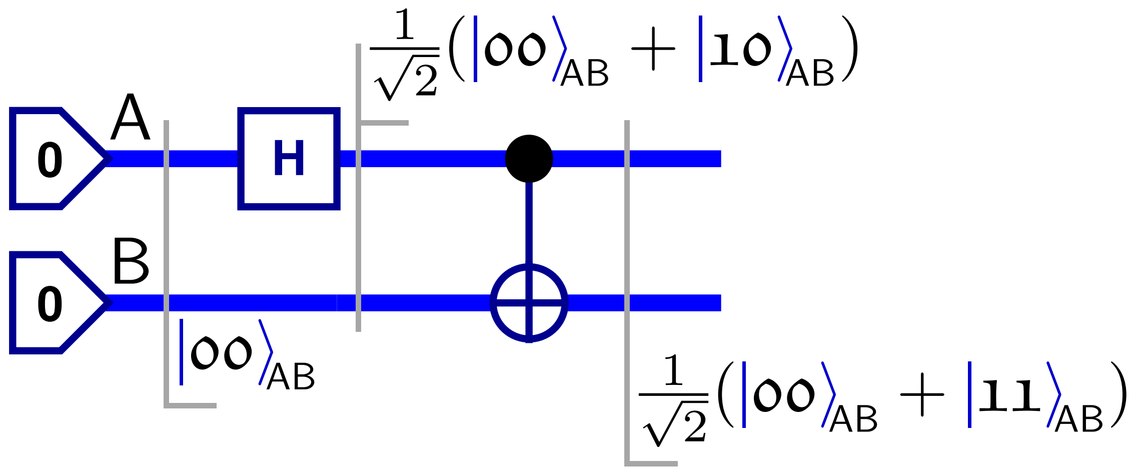

Operators and Kets

- Ket algebra: Rules involving operators.

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  - Operators and kets for disjoint systems commute.

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$$\mathbf{op}^{(\mathsf{X})}|\psi\rangle_{\mathsf{Y}} = |\psi\rangle_{\mathsf{Y}}\mathbf{op}^{(\mathsf{X})}$$

  - Operator multiplication distributes over sums.

$$\mathbf{op}^{(\mathsf{X})}(\alpha|\psi\rangle_{\mathsf{SXY}} + \beta|\phi\rangle_{\mathsf{SXY}}) = \alpha\mathbf{op}^{(\mathsf{X})}|\psi\rangle_{\mathsf{SXY}} + \beta\mathbf{op}^{(\mathsf{X})}|\phi\rangle_{\mathsf{SXY}}$$

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- Ket algebra: Rules involving operators.
  - Operators and kets for disjoint systems commute.

$$\mathbf{op}^{(\mathsf{X})}|\psi\rangle_{\mathsf{Y}} = |\psi\rangle_{\mathsf{Y}}\mathbf{op}^{(\mathsf{X})}$$

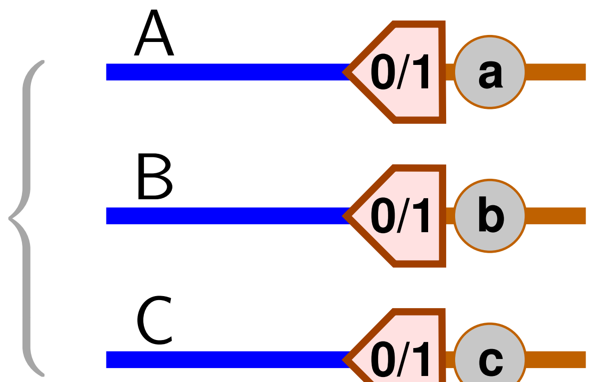
  - Operator multiplication distributes over sums.

$$\mathbf{op}^{(\mathsf{X})}(\alpha|\psi\rangle_{\mathsf{SXY}} + \beta|\phi\rangle_{\mathsf{SXY}}) = \alpha\mathbf{op}^{(\mathsf{X})}|\psi\rangle_{\mathsf{SXY}} + \beta\mathbf{op}^{(\mathsf{X})}|\phi\rangle_{\mathsf{SXY}}$$

  - An operator on S applied to an S-ket expression results in a ket expression using the defined action of the operator.

Measuring Qubits

- Measuring all qubits at once.



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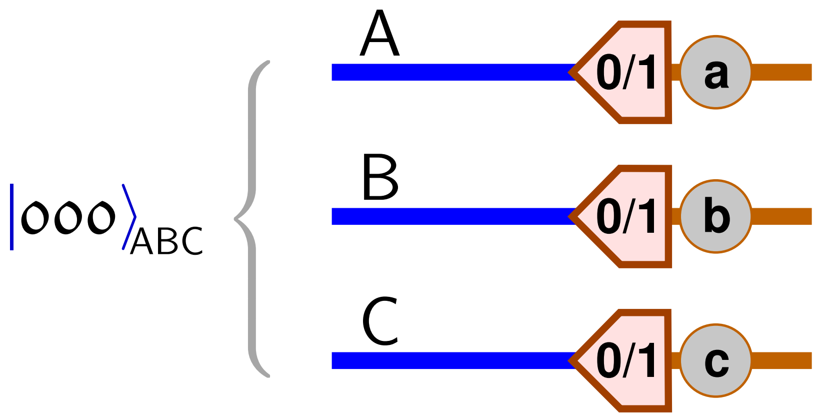

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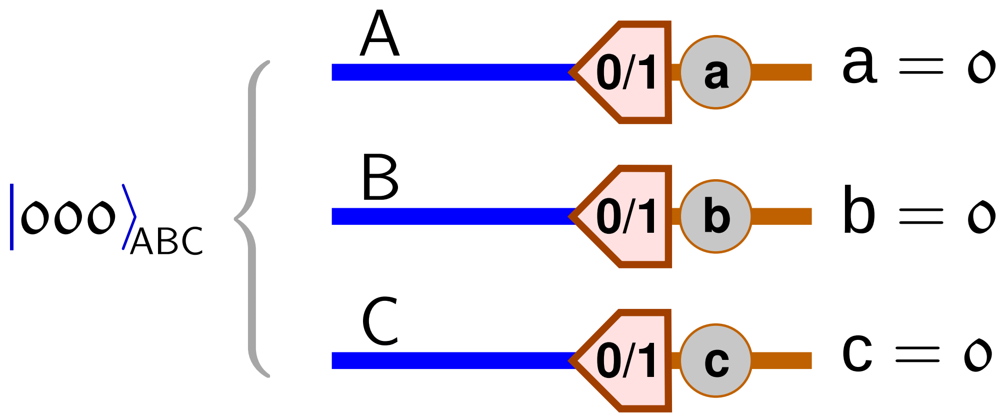

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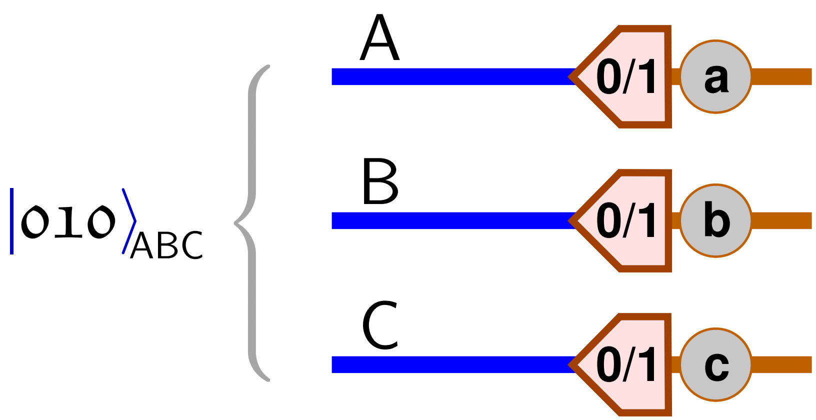

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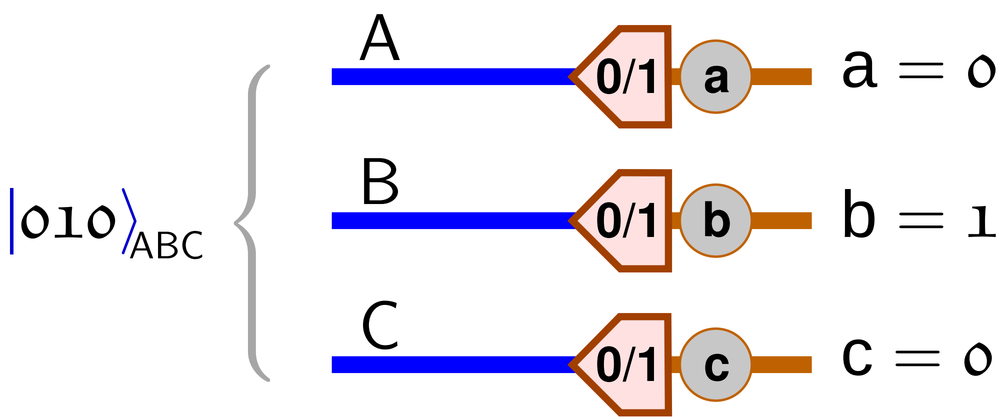

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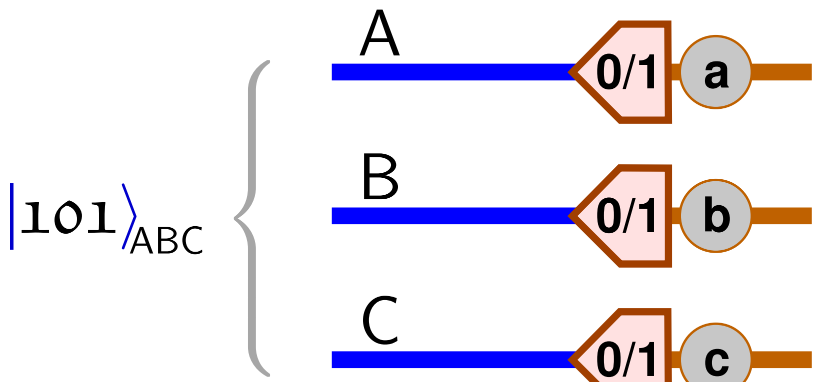

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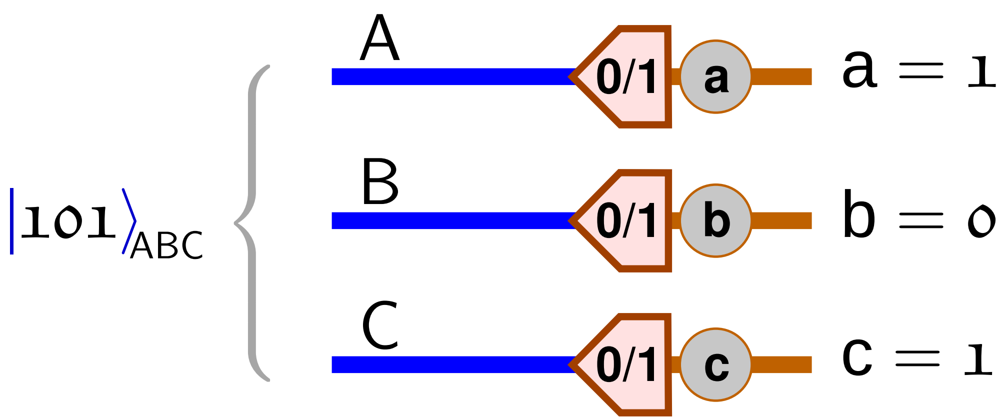

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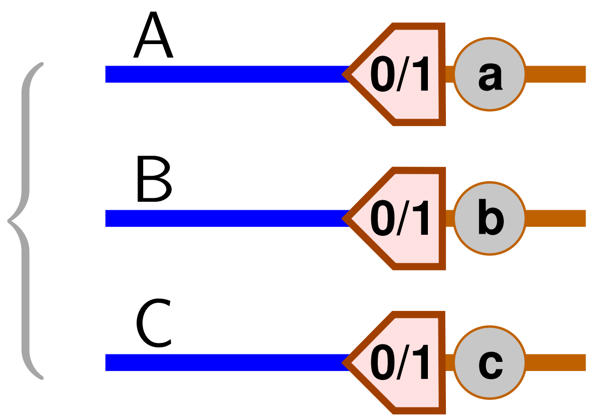
$$\ldots + \alpha_6 |110\rangle_{\mathsf{ABC}} + \ldots$$



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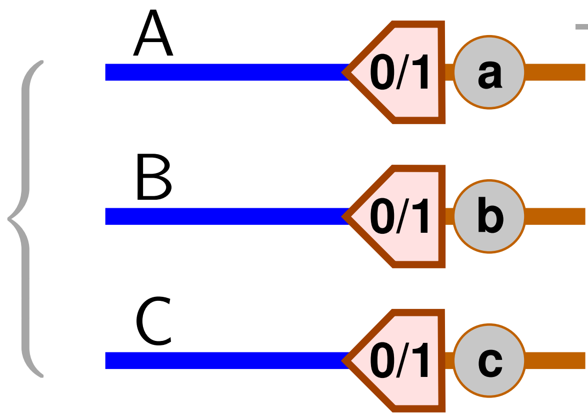

	Probability $\left|\alpha_6\right|^2$	
	$\mathsf{a} = 1$	
$\cdots$	$\mathsf{b} = 1$	$\cdots$
	$\mathsf{c} = 0$	

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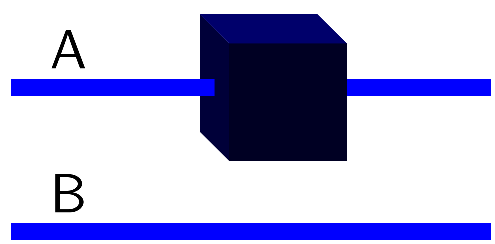

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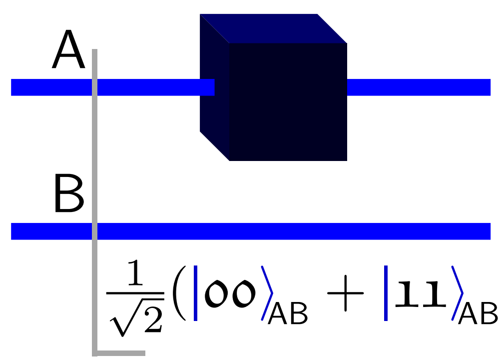

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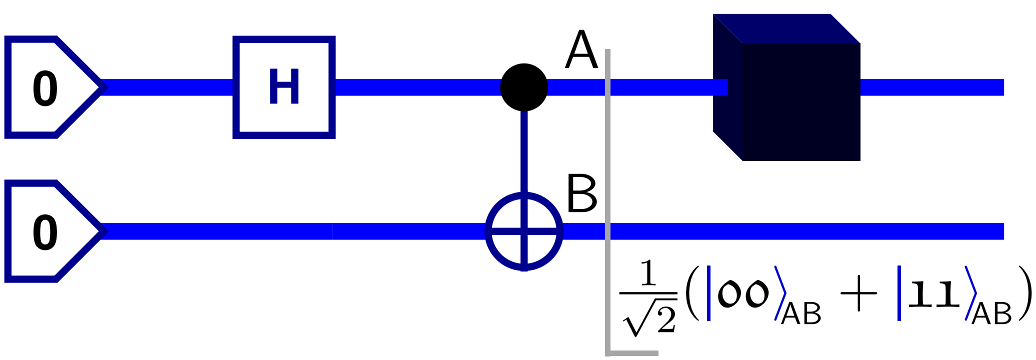

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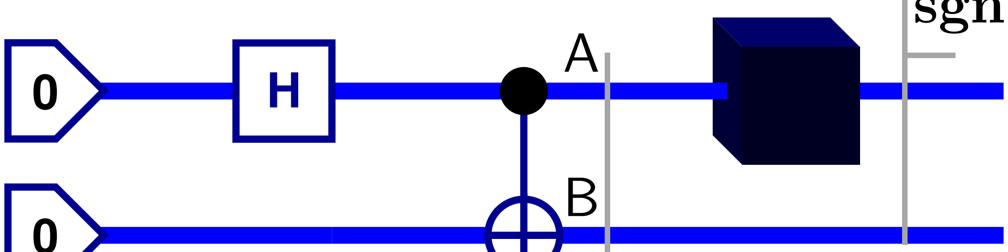

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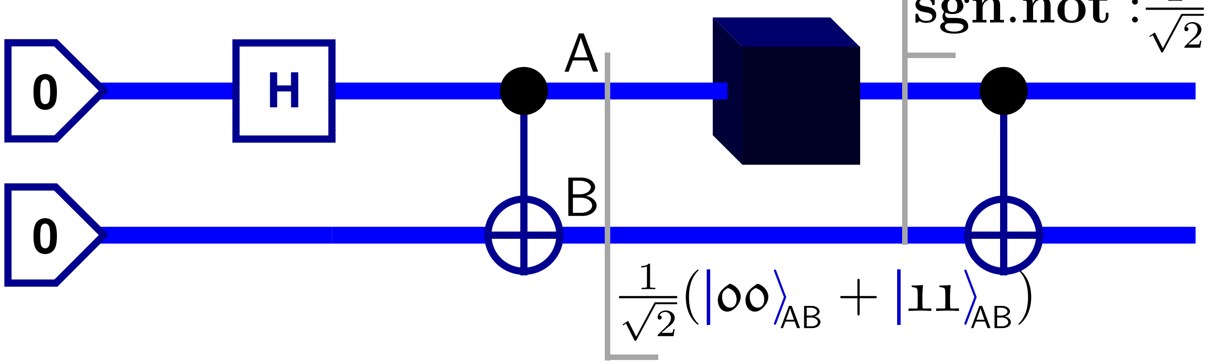

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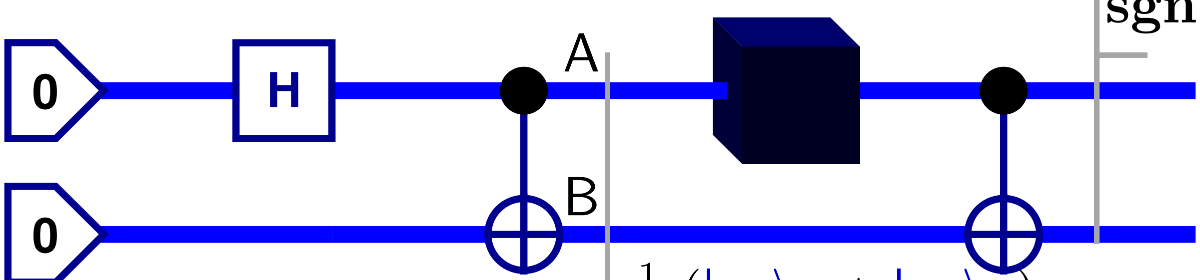

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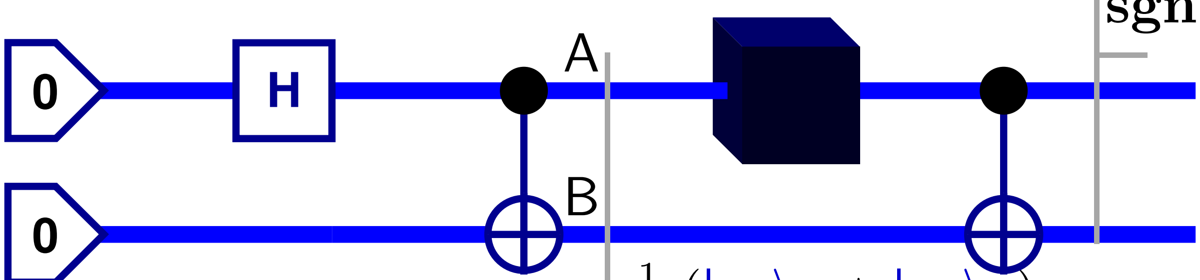

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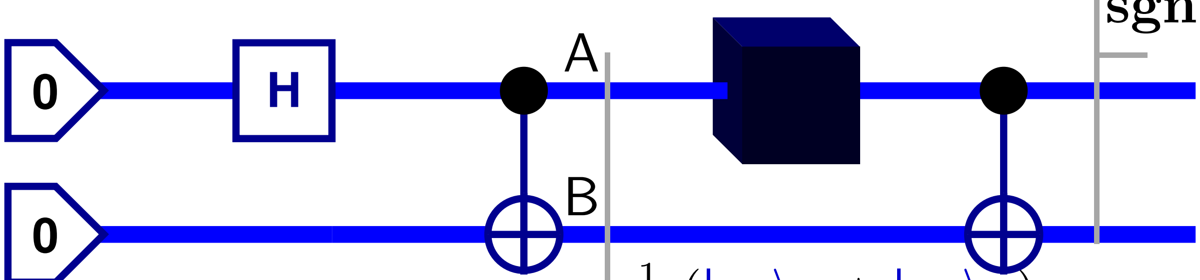

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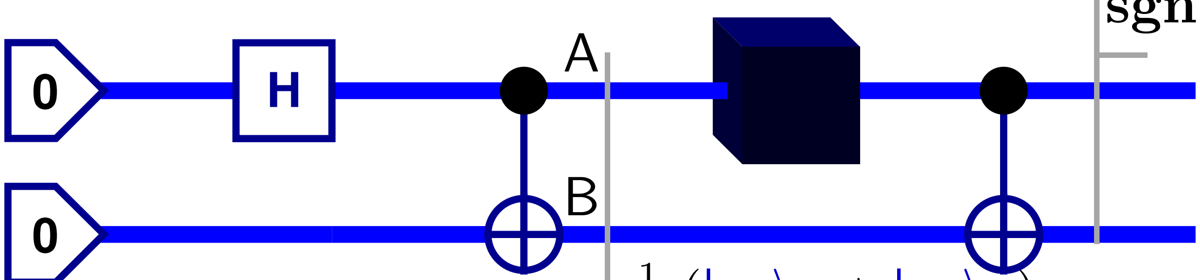

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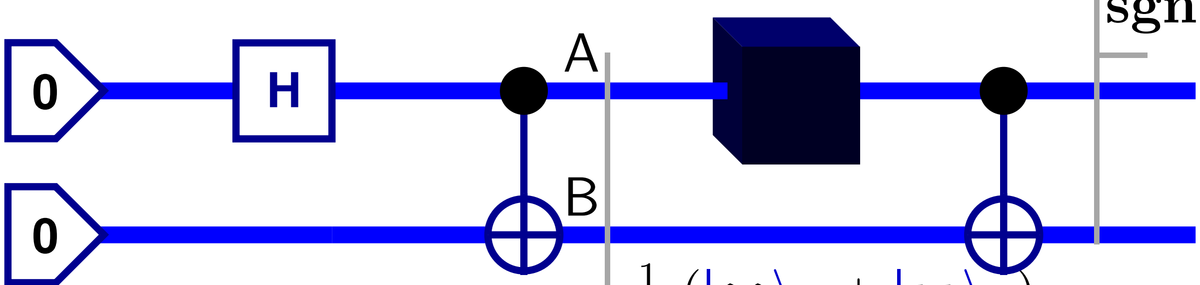

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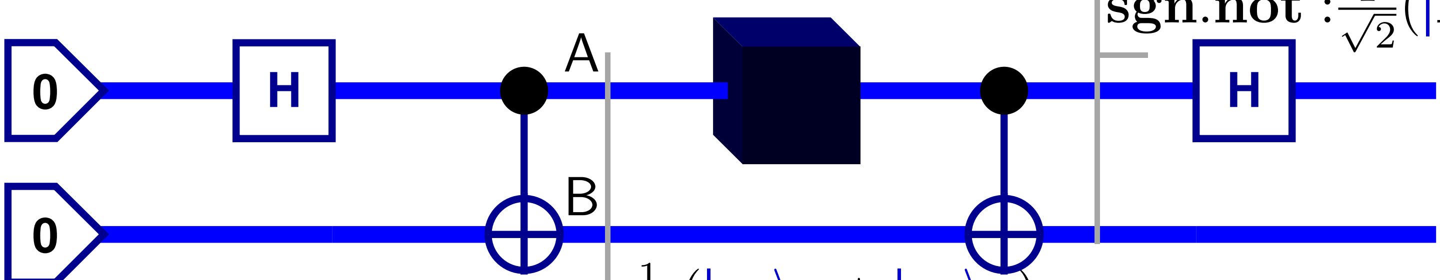

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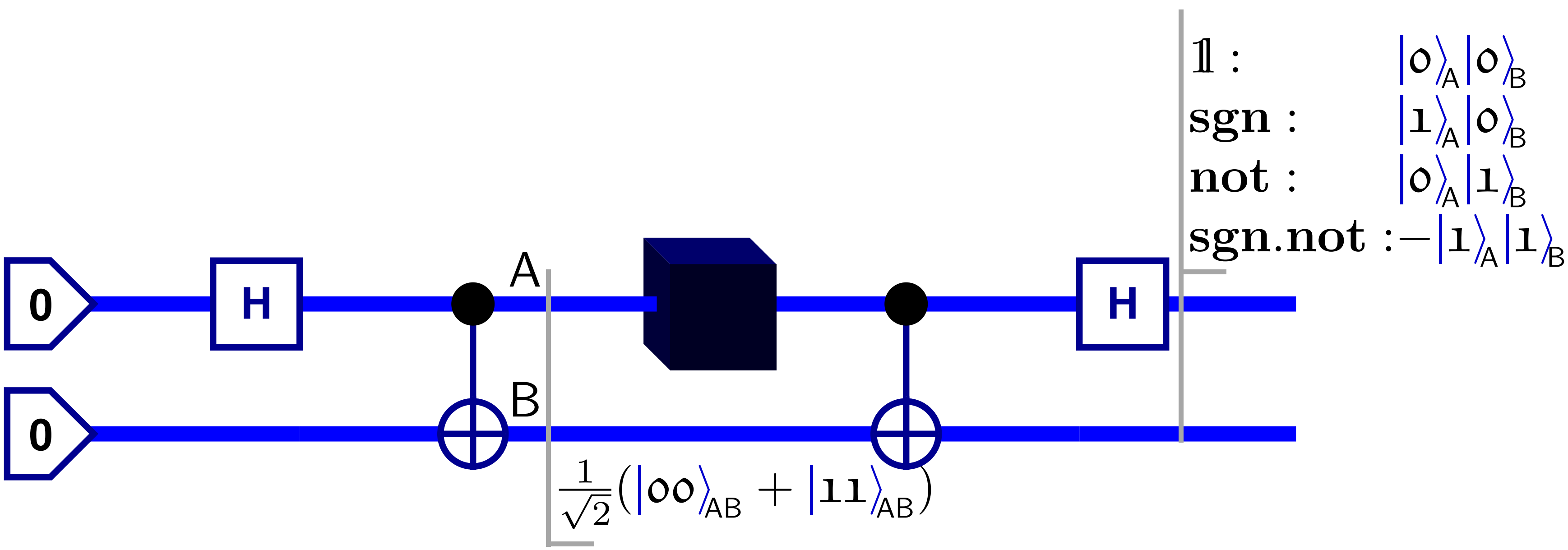

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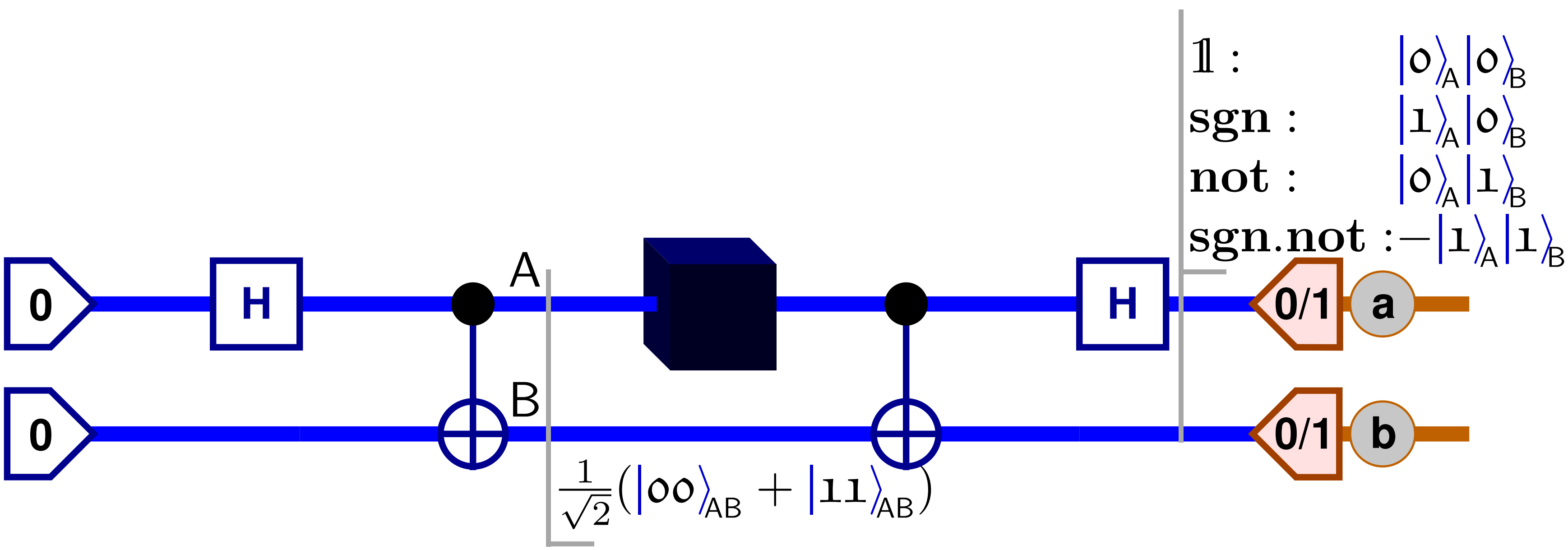

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